THE MATHEMATICS TEACHER

The Official Journal of
The National Council of Teachers of Mathematics
Incorporated 1928

· FEBRUARY · 1930 ·

« Contents »

Volume XXIII Number 2

Published by the
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
Menasha, Wisconsin New York

Application has been made for transfer of second class entry to the post office at Menasha, Wisconsin, under the Act of March 3, 1879. Acceptance for mailing at special rate of postage provided for in Section 1103, Act of October 3, 1917, authorized November 17, 1921.

THE MATHEMATICS TEACHER is published monthly except June, July, August and September. The subscription price is $2.00 per year. Single copies sell at 40 cents each.

THALES

Ancient bust in the Capitoline Museum at Rome, not contemporary with Thales

THE MATHEMATICS TEACHER

Vol. XXIII No. 2

Edited by William David Reeve

The Universality of Mathematics*

By William David Reeve,
Teachers College, Columbia University, New York City

The following news item appeared in the New York *Morning World*, on May 19, 1929:

SURVEY FINDS ARITHMETIC COURSES NEEDLESSLY HARD

Columbia Educational Expert Declares Schools Teach Much More Than the Average Adult Requires

Arithmetic eternal bugbear of the American youngster, is in for a drastic and sweeping revision, if plans made public yesterday at Teachers College, Columbia, are carried into effect. At least 85 per cent more arithmetic is now being taught in our public schools than is required in "life situations," according to the survey.

Coupled with the joy that is due for the school child when he learns that the complicated factoring, involutions and square roots—worst of his "number work"—have been branded by the savants' survey comes the charge that much of this supplemental arithmetic has been injected into the school room by special interests.

"Arithmetic has been shown to be the chief source of non-promotion in the elementary grades," says the inquiry report. "It is too abstruse and is couched in language beyond the years, experience and comprehension of most pupils who are required to study it."

That this report is the first of a series of inquiries on the need of curriculum revision for the different studies in the public schools is intimated in the report. It is pointed out that the method used in exposing the alleged evils of present-day arithmetic teaching "may help in formulating a technique for the solution

* Reprints of this article may be obtained from *The Mathematics Teacher*, 525 W. 120th St., New York City for 10¢ each prepaid.

of other curricular problems." "The revised curricula, it is pointed out, would teach those branches of the subject which are 'now actually in use in adult life.'"

Dr. A. O. Bowden,* education administrator, is largely responsible for the present arithmetic inquiry, carried on under the general auspices of America's greatest teacher mill. Dr. Bowden extended the survey to hundreds of schools in all parts of the nation.

Dr. Bowden lists the seven ways in which the adult would profit from his school study of arithmetic as buying at the store for home consumption, making change, reading newspapers, periodicals and magazines, investing one's savings, aside from one's own business investments; filling out income tax sheets, writing letters, traveling, consulting time tables, recreation, golf, bridge, billiards, etc.

"Probably not more than one-third of all vocations," he says, "such as nursing, sheep raising, street cleaning, etc., require any mathematics beyond the four fundamental operations in arithmetic. That the elementary schools should prepare for all vocations is a belief which has often prevailed and one which has apparently directed much of the practice that has resulted in the overburdening of the present curriculum in arithmetic.

"To discover what the next generation is likely to use, however, is one of the duties of educational sociology. Arithmetic should be taught but not all arithmetic, and we should transmit to the children of tomorrow only that which they are likely to need.

"Some but not much arithmetic is needed by a highly cultured people. The very fact that there have existed most highly cultured people with a knowledge of only the very simplest form of mathematical operations shows that life of a high type can do without much of the complicated mathematics which the school has generally imposed upon the young student. The Greek and Roman systems of mathematics, for example, were extremely simple."

After interpreting arithmetic instruction since the seventeenth century, Dr. Bowden reveals that he sent questionnaires to more than 600 parents all over the country, asking them to check the arithmetic they had used since leaving the elementary schools. Sample problems of each of the eighteen categories or main headings of arithmetic were inclosed.

"Data showed," reports Dr. Bowden, "that only eight out of the eighteen listed are used in large enough amounts to warrant inclusion in a course of study in arithmetic in elementary schools." These eight are:

Numeration and notation, found useful chiefly in reading, United States money, compound denominate numbers, such as dry measure, etc., simple fractions—the most elementary kind, such as $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$; decimal fractions used in dealing with United States money and similar situations; simple operations in percentage, including discounts at the store; allquot parts of 100, such as $12\frac{1}{2}$ equals $\frac{1}{8}$; mensurations, used chiefly in reading.

As is customary with such newspaper reports considerable discussion results and sooner or later two opposing camps arise, one support-

* See Bowden, A. O. Consumer's uses of Arithmetic. Teachers College, Columbia University Bureau of Publications, 1929.

ing the charges set forth in the indictment and the other opposing, if not wholly denying them. Thus, in the New York *Morning World* on May 20, 1929, the following editorial appeared:

THE CURSE OF ARITHMETIC

Even though the discovery is belated, it is gratifying to hear that our educational leaders have finally found out that some 85 per cent more arithmetic is taught in the public schools than is required by "life situations." That is something the rest of us have known for a long, long time, and it has been a perennial cause of wonderment to us that children are compelled to solve problems whose uselessness is little short of fantastic. Which one of these children will ever be called upon to compute how much $3,941.26 will amount to in 7 years, 3 months and 21 days at 6 per cent, with interest compounded semi-annually? Some of them, of course, will have $3,941.26 and may be curious to know what it will amount to in 7 years, 3 months and 21 days. But how they find out is to go to a bank and have a courteous gentleman at a flat-top desk figure it out for them. Similarly with the problem of how much paper, 22 inches wide, will be required to paper a room 18 feet long, 16 feet wide and 9 feet high, due allowance to be made for two windows, 6x3 feet, and three doors, two of them 7x3½ feet and one of them 7x5 feet. That is a job for a paperhanger, if one survives. Similarly with the problem of how many board feet of lumber will be required to build a bridge 132 feet long and 14 feet wide, the planking to be 2 inches thick, the guard rail to be 3 feet high and constructed of 3-inch by 3-inch lumber, mortised and braced as shown in Fig. 37, and the pilings to be 12 inches by 12 inches, drives in clusters of three every 10 feet, as shown in Fig. 38. That is a job for a civil engineer. Similarly with A, B and C and their endless apples. That is a job for the inmate of a booby hatch.

These problems may arise in various fields of human activity. But they do not arise in the life of the average man, and they have no meaning for the average child. Moreover, they are not arithmetic. The processes of arithmetic are four, unless you count the complex processes of involution and evolution that belong more properly to algebra, and once they have been learned, especially in their application to the decimal system and to money, they are all that most of us ever have occasion to use. "Arithmetic," says the report of a survey at Columbia, "has been shown to be the chief source of non-promotion in the elementary grades." And this does not state half of it. It is the chief source of the schoolchild's notion that all study is boredom. If arithmetic were boiled down to its elements for our children, and the reason for learning it carefully explained to them, we lay a wager that their interest in study would be increased manyfold over night.

It is interesting to note the effect that the above news item and editorial from the New York *World* had upon the editor of the New York *Sun*, for on May 21, 1929, there appeared in the *Sun* the following editorial:

LESS AND EASIER MATHEMATICS

Breathes there a schoolboy with soul so dead he is unwilling to shout "Hooray!" over the news that a survey conducted in Teachers College, Columbia, shows that too much arithmetic is taught in the schools? Probably not. If most of the schoolboys, past and present, had had their way there would even be further scaling down of the survey's minimum of profitable arithmetic for the common schools, a minimum which includes numeration and notation, compound denominate numbers, simple fractions, simple exercise in decimal fractions, simple percentage, a few aliquot parts and a bit of mensuration. If mensuration includes the unfathomable mysteries of pi as they affect the magnitude of circles, away with it also!

At the same time it will not be easy for all advocates of easier arithmetic to go all the day with an editorial opinion of the *World*. That newspaper suggests that the way to find out the answer to a complicated problem in compound interest is to get somebody at a bank to do the figuring; to go to a paperhanger for information as to how much paper will be required to cover a given wall space. If that argument is valid why not eliminate addition also from the text books? Are there not adding machines? Is there not somebody who is qualified to answer almost any technical question that can be imagined? The next logical step would be to abolish penmanship because there are typewriters; is even reading essential in these days when the radio comes into almost every home?

The conclusions reached in the survey at Columbia were based largely upon reports from 600 adults that they found little use in practical life for much of the arithmetic they had learned in school. Somehow that also is a little less than convincing. If that is a fair test there is little reason for studying Latin, rhetoric, geology, geography, physiology and a dozen other things. It would be interesting to know how many persons now in middle age and in business remember to care what the medulla oblongata is, whether a synechdoche is a figure of speech or a figure of geometry, whether the Gobi Desert is in Mongolia or in the moon. When a practical man thinks of the time he wasted scanning iambic pentameters in school, when he hasn't seen one in his counting house for years, it is enough to goad him to despair.

Yet the professors might as well know now as later that all of them will soon be out of jobs if nothing is to be taught in the schools which will not come in handy in the adult's daily struggle for his daily bread. Let them make arithmetic easier if they will and can. Then let them cut down on grammar, logic, civics, history and all the rest of it. How does it help a man to know the date when Horsa and Hengist occupied the Isles of Thanet when that man's immediate job is selling a consignment of grindstones to a country merchant in Manitoba? By the time these matters have been settled to the satisfaction of everybody concerned it should be almost possible to do away with schooling altogether.

Another newspaper editor didn't think much of the conclusions credited to Dr. Bowden. According to the *Teachers College Record* for December, 1929, the Columbus (Ohio) *Dispatch* had the following to say:

One of the "educational experts" of Teachers College, in Columbia University, says of the study of arithmetic: "Arithmetic should be taught, but not all arithmetic. We should transmit to the children of tomorrow only that which they are likely to need. Even highly cultured people have need for only a small amount of mathematics."

His conclusion is that something like 85 per cent of the arithmetic usually taught in the schools should be dropped out. He can see little use for anything more than simple addition, subtraction, multiplication, and division. A questionnaire sent to some 600 persons found but few to reply that they had found the higher mathematics of any direct benefit to them.

This was quite natural, as a result of the questionnaire method. One would like to see the results of a questionnaire sent to 100,000 or more men and women who have been "questionnaired" for the past 20 years asking th one simple question, "Do you favor the questonnaire, as a means of getting comprehensive, accurate and reliable information on debatable matters of opinion or sentiment?"

The difficulty with this Columbia expert's opinion as to the teaching of mathematics is that its practical application in the schools would leave a countless number of young people handicapped, later on, when they desired to push into one or another interesting and profitable life work, or subject of study, and found that they did not possess the mathematical key necessary to unlock the door. It is the place of education to open up, not close, the gateways of a larger life.

As a matter of fact many of the topics which were denounced by the *World* editorials referred to above are no longer taught in the better schools.

It should not hurt arithmetic among people who discriminate if someone says that 85 per cent of the arithmetic that has already been discarded by progressive teachers is worthless. However, when the critics begin to assail the obsolete some educators are swept off their feet and some of the more valuable parts of mathematics may be lost to the pupil. It will be worth while for us to see what some of these more valuable parts are.

If I, or any other teacher of mathematics, attempted to show why arithmetic shought be taught, many people would say "What can you expect? He teaches mathematics. Of course he sees value in it." That is one reason why in this discussion I prefer to quote at length the opinions of prominent people outside the field of mathematics as well as to set forth a few opinions of those of us who have had the interest of the subject at heart and who because of this interest are often accused of being unduly prejudiced.

This discussion is not intended in any sense as a defense of mathematics. At its best the subject needs no defense. I wish, however, to point out the need for improvement in the selection and arrange-

ment of content and in methods of teaching mathematics because I believe that most of the criticisms if properly understood would be shown to be criticisms aimed mostly at faulty methods of teaching more than anything else.

Recent discussions pro and con relative to the proper place of arithmetic and other mathematical subjects in the course of study have revealed an astonishing amount of ignorance with reference to the significance of number. We Americans are so used to taking so many things for granted that it is not altogether surprising to find a thoughtless and ill informed person claiming that one can teach everything one needs to know about the whole field of mathematics in a few lessons. Even a distinguished scientist is reported to have said that one can teach a boy all the mathematics he needs to know to understand physics in thirty minutes. I once taught mathematics in the high school where the son of this scientist studied his physics and I was told that it took this son of a famous father longer than thirty minutes to master the mathematics necessary to undersand the physics taught in that school. Moreover, it will take several minutes to explain one algebraic formula in a footnote of a page of a high school physics text written by this same scientist.

Is it surprising then that we hear some curriculum maker saying that if one wants to know what arithmetic to teach in the schools, he should make a survey of the community and ask adult members of that community how they use arithmetic in their daily lives? There are two apparent weaknesses with this method of procedure. In the first place people do not know how they use arithmetic and secondly, they can not tell what use they might make of number if they knew more about it; and so for other parts of mathematics. Professor Judd[1] thinks that a large part of the trouble is due to the fact that arithmetic and reading have been traditionally looked upon as "tool subjects." He speculates very interestingly as to what might have been the probable "effect on subsequent educational thinking" if, for example, reading and arithmetic had been originally referred to as the "right and left hands of learning," instead of "tool subjects." He says:

[1] Judd, C. H. "The Fallacy of Treating School Subjects as Tool Subjects." *Third Year Book, National Council of Teachers of Mathematics.* Page 1. Bureau of Publications, Teachers College, Columbia University.

See also Judd, C. H. "Informational Versus Computational Mathematics." *The Mathematics Teacher,* 22: 187-196 for one of the most stimulating discussions available.

When a writer on education says arithmetic is a "tool subject," what does he mean? We have only to appeal to present-day educational literature to secure a clear answer. Arithmetic is thought of as a subject which is a kind of necessary evil. If one wants to pry loose interest from capital, one will have to do a little calculating. If one wants to purchase bricks and lumber for the highly important business of building a house, one will have to delay in the truly constructive work long enough to add up a column of figures. If one is to engage in the interesting occupation of compounding a recipe, one will have to stop and measure and count the ingredients. The ends of life are possessions, and buildings, and food; these are the matters with which many educators declare themselves to be concerned; these, they tell us, are the substantial sources of valuable mental experience; these are the concrete facts of life. Number is to the minds of many of our contemporaries a formal abstract something which has crept into the schools and crowded out property, and buildings, and food. Ardent critics of mathematics would have us believe that number is a tool which one uses perhaps twice a day, possibly three times, but never without apologies and never without a feeling of aversion at the delay which number thinking entails. The popular plea today is a plea for reduction all along the line. Let us find out how frequently the people of Iowa and Boston really use arithmetic in their daily lives, let us be rid of antique puzzles about ditch digging and the rest, let us laugh algebra and geometry out of the high school.

In short the common weakness of our present thinking and practice is to overemphasize everything that is concrete as can be seen by even a cursory examination of some of our current textbooks where such emphasis is given to concreteness that we have all pictures and no arithmetic, all political science and no arithmetic, all project and no arithmetic except a type that obscures the fact that "number is an ever guiding principle of life." Such a point of view ignores the idea that number system has consciously or unconsciously changed your life and mine because it is a way of thinking just as algebra is. Such an attitude reverses the scriptural passage so often quoted and says that the things that are seen are eternal. To such folk "number is merely something to be used now and then as a vague and unsatisfactory substitute for things that are concrete and substantial and truly important."

To quote Professor Judd again:

The pupil who is drilled day after day in the use of number is acquiring a mode of thought which will change all of his later mental operations. The individual who has had experience with number is no longer capable of returning to the level of loose, inexact thinking that characterized his earlier methods of viewing the world. The curriculum maker who would be true to the facts must inquire into the subtle modes of thinking which lie deeper than the mere expressions of number in the ordinary meaning of that word.

In spite of the preceding statement it is a difficult thing for the ordinary person to appreciate what it has cost to bring our number system up to its present standard of excellence. We often forget, if we have ever known it at all, that the race has taken a long time to develop its present ideas of honesty and accuracy in measurement. Many of our moderns are woefully ignorant of the importance of precision in our present day civilization partly because they do not know the historical facts connected with the development from the loose, unwieldly, vague ideas of precision held by primitive man to our present "universal recognition of the importance of the idea of precision."

If one has any doubt the clumsiness of some of the previous attempts of the race to express itself in matters of precision let him try to carry on the fundamental operations by means of the old Roman numerals. Thus, "This idea of precision is not a collection of number names, nor a series of addition combinations taught in subtraction; it is a general idea refined out of many uses of number and ultimately taking its place in the minds of men as a guiding principle dominating every train of thought that a modern man follows."

However, trouble arises because some of our modern curriculum builders do not realize all of the implications of the preceding statement. Witness the present dearth of material which can be used to enlighten one with respect to the explicit or implicit use of the idea of precision. Let me quote Professor Judd once more:

> I confess I am astonished when I see some of the analyses which purport to be the scientific foundations on which school curriculums are to be built and find no mention of these general ideas of order and arrangement and precision. I am told that the school should teach children how to make change and how to measure wall paper and how to tell time and that sections of arithmetic should be devoted to these specific tasks, but I look in vain for any appreciation of the fact that the school ought to lead pupils who have only a hazy and unsystematic notion of the world to see the value of arrangement and order in all thinking and to cultivate the general ideas of regularity and precision.
>
> I venture the prophecy that we are just at the point where we are about to leave behind the inadequate psychology which has in recent years taught that mental life is a bundle of particular ideas. We shall realize that general ideas do not arise in the untrained mind but are the highest products of constructive thinking. The number system, the mathematical formulas of algebra and geometry, however inefficiently taught in the past, have helped the race to organize and arrange the world in which we live. He who thinks of the number scheme as a trivial addendum to the mind's equipment does not know his history.

One reason why we find ourselves in our present embarrassment is that our teaching is at fault. In my judgment criticism which should have been directed against poor teaching has in many cases been leveled at the subject instead. Many teachers have looked upon the number system as computational arithmetic entirely instead of a combination of computational with informational arithmetic. In other words, "Teachers have not regarded arithmetic as a science made up of general modes of thinking; they have thought of it as a series of rules."

This attitude on the part of the teacher is not to be wondered at if we recall the familiar fact that computational arithmetic is relatively easy to teach as compared with a real understanding of number. In the same way our past methods of teaching algebra have resulted in a kind of algebraic procedure that is mechanistic at best.

The large number of failures in arithmetic and algebra especially is evidence of a lack of expertness in our teaching of these subjects. Other things being equal, the further up we go in the grades the worse the failures become.

Speaking on *The Reorganization of Secondary School Mathematics* Mr. Betz[2] a well known classroom teacher of mathematics said:

> The world in which we live is incurably mathematical. Every human being is born into a physical universe in which quantity, shape, and size play an indispensible rôle. The geometric principles of equality, symmetry, congruence, and similarity are implanted in the very nature of things. It is apparent, for example, that we cannot make or manufacture the simplest article without giving the constant attention to its form, its dimensions, and the proper relation of its parts. The art of measurement permeates the fabric of modern civilization at every point. It underlies all applied work in engineering, technology, and manufacturing. Without measurement and computation the world of science would cease. Algebra furnishes economical methods and formulas for many of these computations. Trigonometry, being essentially the art of indirect measurement, forms the necessary background for the making of maps and survey plans of all sorts. It underlies the art of navigation. Our entire civilization, our sciences, our modes of thinking, have a mathematical core.

According to one of the Denver (Colorado) papers last June, "Myron C. Taylor, most widely known as the successor to Judge Gary" in an address at Colgate University had the following to say:

> What shall we put into the mind? For the uses of the present day we could hardly put too much mathematics into it. A brain that is mathematically minded

[2] Betz, Wm. "The Reorganization of Secondary School Mathematics," *Progressive Education,* 5 page 377, 1928.

has a decided advantage over any other in affairs of trade and commerce; it is a fine type of mind to approach problems in physics, and it is just the sort of mind that would lead one to be philosophical. And so perhaps if one were permitted to study only one subject, depending upon companionship and one's every-day contacts with the world to pick up the language of the community after a fashion, mathematics would serve one better than any other subject in the course of study. So it would seem that one cannot give too much attention to mathematics.

Professor Birkhoff[3] of Harvard recently wrote the following:

Nearly 2500 years ago the Greek philosopher and mathematician Pythagoras conjectured "that all things are numbers." His bold affirmation was based upon a slight knowledge of the heavenly bodies, of the regular geometric solids, and of the stretched musical string. Yet all physical science since his day has attested the truth of his magnificent generalization by establishing the universality of numerical law. By adopting the Pythagorean outlook and applying it to our own inconceivably more body of knowledge we too may hope to obtain new enlightenment as to what science suggests.

It is gratifying to find a friend for mathematics in the editor of *The Saturday Evening Post* who said in the *Post* for June 1, 1929:

THE KEY OF THE UNIVERSE

Mathematics, if we are not greatly mistaken, is presently destined to play a much larger part in our general scheme of education than it ever has in the past. One is forced to this conclusion not by the insistent demands of students but by the consideration that the tools and the methods offered by this science have been so largely responsible for the extraordinary advances in other sciences which the past generation has witnessed. The more mathematics contributes to the development of other sciences the more dependent upon it they become.

This state of affairs has come to such a pass that already the layman who wishes to keep up with modern thought is restricted to the use of the most elementary books, for if he goes to the more advanced works, the sources of really sound and substantial information, he at once finds his progress barred by an entanglement of calculus and other branches of higher mathematics which he either has not studied or has completely forgotten.

In electricity, physics, chemistry and astronomy the educated reader might expect to encounter this difficulty, but he may be surprised to find that he will be no better off if he attacks biology, physiology or any serious discussion of the structure of matter, of the nature of the atom, or of modern conceptions of time, space and what we call the universe. It is as if a new and untranslatable language had suddenly come into common use among our real intellectuals and we must forever remain cut off from knowledge of the amazing and thrilling doings in the learned world about us, simply because they can only be described in that

[3] Birkhoff, George D. Science and Spiritual Perspective—A New Philosophy. *Century Magazine,* June, 1929.

language—a strange tongue which it would take half a lifetime to master and which no man can render into plain understandable English.

We hark back to the sixteenth century as having been one of the most colorful in all history, because all the Old World was agog with the discovery of the New. For a hundred years the wild tales and rumors which drifted eastward across the Atlantic and were talked over and retold by round-eyed burghers in every inn and alehouse made high and low feel that they were living in a continuous fairy tale full of wonders, monsters and strange happenings. It was, in truth, a mighty century; and yet there are those who, without belittling the achievements of the early voyagers, will declare that the discovery of the land beyond the sea was as nothing when set alongside certain recent explorations of our boundless but finite universe and the discovery of the countless little universes which are as numerous as the atoms themselves. One difference is that though we laymen suspect that something of supreme importance has been going on around us, and epoch-making discoveries have been made, our Columbuses and Einsteins have not brought back from the jumping-off places of the universe any golden nuggets or aromatic spices or exotic living creasures as understandable trophies of their voyages. About all we can learn from the lofty and slippery wall of ignorance which confronts us is the conviction that we know little or nothing about time or space or even the electricity which we buy so freely at a few cents a kilowatt hour.

There is, of course, a practical answer which should perhaps silence these lamentations of ignorance. We shall be told that if our modest intellectual equipment causes us to funk these higher jumps which Mr. Einstein and other scientists have set for us, there are plenty of others within our powers. There is always bridge or contract or even chess; and every newspaper has its syndicated crossword puzzles and so-called brain teasers.

This vigorous snub should perhaps put in their places those humble learners who want to learn for the sheer thrill and pleasure of learning and knowing; but it does not answer those practical souls who are beginning to realize the extent to which mathematics is invading every field of everyday life. It does not answer the youngster who wants to build bridges or skyscrapers and knows that he must mix higher mathematics with his mortar, build it into his foundations, and use it to test every I beam and angle iron that goes into his work. It does not answer the young research physician or biologist or chemist or industrial engineer who knows that he cannot reach the top of his profession owing to his lack of advanced mathematical knowledge. It does not answer him who has discovered, or thinks he has discovered, that higher mathematics is the master key of the universe.

There are still those who suppose that this untranslatable language we call mathematics is as dead as classical Latin or Sanskrit. They have no one to tell them that it is as truly a living science as physics or chemistry; that new and startling advances in it are still being made every few years, and that with every day that passes it becomes more and more interwoven with our daily lives and with familiar things of which we make daily use. Industry recognizes, in a measure, its debt to this science. Our larger electrical and telephone companies have fostered it consistently, and in recent years farsighted insurance companies have thought it worth their while to support mathematical research.

Every circumstance points to the belief that if we are to keep up with the procession of human progress our schools and colleges will have to devote more time to this subject, offer more advanced courses and stress their importance to every student who hopes to make any real progress in the physical sciences.

Finally let us recall what one of the most noted teachers of mathematics in this country has to say about mathematics. If more teachers of mathematics could catch a little more of his enthusiasm and interest for the subject which they have chosen to represent our critics would have less about which to complain. David Eugene Smith[4] once said:

One thing that mathematics early imparts, unless hindered from so doing, is the idea that here, at last, is an immortality that is seemingly tangible,—the immortality of a mathematical law. The student of algebra, for example, may well question the use of the traditional curriculum, but when he finds the value of $(a+b)^2$ he has come in contact with an eternal law. The laws of the Medes and Persians, unchangeable though they were thought to be, have all perished; the canons that bound Egyptian activities for thousands of years exist only in the ancient records, preserved in our museums of antiquity; the laws of Rome, which at one time dominated the legal world, have given place to modern codes; and the laws that we make today are certain to be changed tomorrow. But in the midst of all these changes it has ever been true, it is true today, it shall be true in all the future of this earth, and it is equally true throughout the universe whether in the algebra of Flatland or in that of the space in which we live, that $(a+b)^2 = a^2 + 2ab + b^2$.

We may change the symbols,—they are temporary expedients to convey the idea; we may speak in different tongues,—they are local expedients to convey thought; but it is inconceivable to us that the relation which the formula expresses should not be true always and everywhere,—a tangible symbol of the immortality of law.

What I learned in chemistry, as a boy, seemed true at the time, but much of it today is known to be false. What I learn of molecular physics seems at the present time like children's stories, interesting but puerile. What we learn in history may be true in some degree, but is certain to be false in many particulars. So we may run the gamut of learning, and nowhere, save in mathematics alone, do we find that which stands as a tangible symbol of the immortality of law, true "yesterday, today, and forever."

But does the teacher make this known to the student? Does the student come to feel the significance of this fact,—this fact so full of awe to the normal mind, this evidence of immortality that never comes to his consciousness until he meets it in mathematics? I do not know, nor do I know how much else that is great, that has tremendous significance, is taught or is not taught in science, in letters, in history, or in art. I only know that mathematics can do this thing; that it can (and it should) give, to the degree that the pupil is able to receive it,

[4] Smith, David Eugene. "Religio Mathematici." *American Mathematical Monthly*, October, 1921.

the idea that before the world was created, before our solar system was formed, and after our system shall cease to be, the everyday laws of mathmatics stood and shall stand for immortal truth,—for laws that are divine in their infinite endurance. It is not necessary, it is not desirable, that we should preach these things in the halls of learning; but it is essential that we should feel their significance. This done and all the rest "shall be added unto you." Stated in another way, the immortality of law means that we come in touch with the invariant. The tyro in mathematics comes early upon the invariant properties of a figure as seen in the theory of elementary projection. In a wider sense, however, all geometry is a science of invariance. We prove a law for a general plane triangle and it never varies, whatever we do to the figure. If we prove that $a^2 = b^2 + c^2 - 2bc \cos A$, then, however A may change, the law itself will never vary. In it the pupil comes into touch with the unchangeable, with the absolute.

It is the same with all other laws of geometry. In any convex polyhedron, whatever its shape, the law remains that the number of faces plus the number of vertices is equal to the number of edges increased by two.

"Change and decay in all around I see," but the established properties of a general geometric figure, in our space, are as unchangeable in that space as divinity itself. Stated in still another way, the immortality of law and the invariability of mathematical principles mean the eternity of mathematics. To come into relation with a science which was illustrated by the spiral nebulae before our solar system was formed, which only now reveals to us those laws of crystals which were in operation long before life appeared upon the earth, and which is also entirely independent of matter, so that if we could imagine the universe destroyed absolutely, the laws would still be true,—to come into relation with such a science makes real to us, as no other discipline in our curriculum can possibly do, the ideal of truth eternal.

OUR INFINITESIMAL NATURE

I know of nothing which acts as such a powerful antidote to that which I ventured to call "opinionatedness," as a study of mathematics. To know that the light from solar systems far larger than our own has been thousands of years in reaching us, gives us an idea of our infinitesimal nature, in comparison with space about us, that can come only with a study of the science that it is ours to teach. A bacillus in our veins, so small as to be invisible through a powerful microscope, is a giant compared with ourselves in our relation to this space in which we live. Our doubts, our beliefs, our hopes, our fears are all so trivial, so infinitesimal, so like a lost electron in our solar system, as compared with our relative importance in the universe as revealed to us by the calculations which mathematics brings to bear upon the great problem! Cowper wrote well when he put in verse the words,

> "God never meant that man should scale the heavens
> By strides of human wisdom,"

and even the mathematics of youth confirms the thought.

THALES

The First of The Seven Wise Men of Greece

THE FIRST of the Greeks to take any scientific interest in mathematics in general, and in the union of astronomy, geometry, and the theory of numbers in particular, was Thales[1] an ancient bust of whom, like that in the Capitoline Museum at Rome, appears as the frontispiece in this issue. Before his time there had been the usual interest of early peoples in the mystery of the heavens, as witness the statement of the poet Archilochus that a solar eclipse was observed some time before Thales was born. Not until the time of Thales, however, did the science of mathematics begin the Greek civilization.

Miletus was then a trading and colonizing center, a city of wealth and influence. Herodotus (c. 450 B.C.), tells us that Thales was of Phoenician descent; but his mother, Cleobuline, bore a Greek name, while the name of his father, Examius, is Carian. The name of Thales himself was probably a common one.

Thales was a merchant in his younger days, a statesman in his middle life, and a mathematician, astronomer, and philosopher in his later years. In his mercantile ventures he seems to have been unusually successful, even in dealing with the shrewdest of the Greek trading races. Aristotle (c. 340 B.C.) tells us how he secured control of all the oil presses in Miletus and Chios in a year when olives promised to be plentiful, subletting them at his own rental when the season came.

Trade was then an honorable calling, and Thales seems to have traveled in Egypt on his commercial ventures, and early writers tell of his also visiting both Crete and Asia. He was not the only mathematician to have thus turned trade to profit, for Plutarch has this to say of him: "Some report that Thales and Hippocrates the mathematician traded, and that Plato defrayed the charges of his travels by selling oil in Egypt." In this way Thales may have accumulated the wealth that permitted him to indulge his taste for learning and to found the Ionian School. It was through this indulgence that he

[1] The material for this historical sketch can be found in Smith's History of Mathematics, Vol. I, pp. 64-68.

acquired such a reputation as to be enrolled as the first among the Seven Wise Men of Greece, and that he was esteemed as the father of Greek astronomy, geometry, and arithmetic.

Of the nature of the arithmetic that Thales brought back from Egypt we have little direct knowledge. Iamblichus of Chalcis (c. 325 A.D.) tells us that he defined number as a system of units, and adds that this definition and that of unity came from Egypt. This is not much, but it is enough to show that Thales was interested in something besides the merely practical. It is probable that he knew many other number relations, for the Ahmes papyrus contains some work in progressions, and such knowledge would hardly escape so careful an observer as Thales. It is, however, in his work in founding deductive geometry and in his capacity as a teacher of Pythagoras rather than as a discoverer of facts that Thales commands our attention.

He took much interest in astronomy, and Herodotus (I, 74) tells us that he even succeeded in predicting an eclipse. Some authorities supposed this eclipse to have occurred on May 28, 585 B.C., while others place it about twenty-five years earlier. He could have obtained certain information on this subject from a study of the Chaldean records, but whether this was his source of information we cannot say. At the present time we have numerous cuneiform tablets of the 7th century B.C. which record such prognostications. One of these reads: "To the king, my master, I have written that there was about to be an eclipse. The eclipse has now taken place. This is a sign of peace for the king, my master."

A man like Thales, possessed of an inquisitive mind, coming in contact with scholars from other lands, either on his travels or in the commercial center of Miletus, would lose no opportunity to secure information of this kind and to make use of it in his teaching. Doubtless his scientific training led him to discard the astrological notions of the Chaldeans but to retain whatever of astronomy came to his attention.

In geometry he is credited with a few of the simplest propositions relating to plane figures. The list, according to the most reliable ancient writers, is as follows:

1. *Any circle is bisected by its diameter.*
2. *The angles at the base of an isosceles triangle are equal.*
3. *When two lines intersect, the vertical angles are equal.*

4. *An angle in a semicircle is a right angle.*

5. *The sides of a similar traingles are proportional.*

6. *Two triangles are congruent if they have two angles and a side respectively equal. (Euclid, I, 26.)*

As propositions in geometry these may seem trivial, since they are intuitive statements; but their very simplicity leads us to believe that it was the fact that Thales was the first to prove them that led Eudemus (c. 335 B.C.) and other early writers to mention them. Up to this time geometry had been confined almost exclusively to the measurement of surfaces and solids, and the great contribution of Thales lay in suggesting a geometry of limes and in making the subject abstract With him we first meet with the idea of a logical proof as applied to geometry, and it is for this reason that he is looked upon, and properly so, as one of the great founders of mathematical science. In the history of mathematics, as in the history of civilization in general, it is the setting forth of a great idea that counts. Without Thales there would not have been a Pythagoras—or such a Pythagoras; and without Pythagoras there would not have been a Plato—or such a Plato.

In philosophy he is said to have asserted that water is the origin of all things, that everything is filled with gods, that the soul is that which originates motion, and that matter is infinitely divisible; but his basis for belief in these assertions is not very satisfactory. Like most of his contemporaries, he left no written works.

NOTICE TO SUBSCRIBERS

Every member of the National Council of Teachers of Mathematics is entitled to vote for the officers who are to be chosen for the ensuing year at the next annual meeting on Fbruary 21, 1930 at Atlantic City. Those members who do not intend to vote at the meeting on February 21 should mark the "Official Ballot" on page 62 of the January issue of THE MATHEMATICS TEACHER and send it to the Secretary, Edwin W. Schreiber, 434 W. Adams St., Macomb, Illinois, at once. No votes can be counted that are received after that date.

Proposed Syllabus in Plane and Solid Geometry

By George W. Evans
107 Ocean Street
Lynn, Massachusetts

The following list of propositions, 104 in number, have for plane and solid geometry the same total content as the list of 181 that constitutes the College Entrance Examination Board's definition of those two subjects. It is based upon measurement, and uses systematic approximation as a means of dealing with irrationals. Algebra and trigonometry are used wherever they contribute to clearness and brevity.

The propositions are divided into successive groups so that in each the student may realize that there is an immediate aim for the group, as well as a general aim towards which each group makes obvious progress.

The *Assumptions* listed in the beginning are those usually made, whether tacitly or not, in current American text books.

Plane Figures

Assumptions.

1. Two straight lines in the same plane meet if at all in one point only.
2. Not more than one parallel can be drawn to a line from a point without.
3. Not more than one perpendicular can be drawn to a line at a point in it.
4. Superposition is the test of congruence.
5. Overturning a figure reverses its order of arrangement.
6. Figures are equal when their measurement numbers are equal. All congruent figures are equal, but figures may be equal without being congruent.
7. If in any two circles the central distance is less than the sum of the radii, and greater than their difference, the circles will meet in

two points only, and those points will be on opposite sides of the line of centers.

8. Of lines having the same extremities the straight line is the shortest.

I. Measuring Distances and Angles.

P1. There cannot be two different numbers having the same converging approximations.

P2. In any circle, if two central angles are equal, the arcs they intercept are equal.

P3. In any circle, if two arcs are equal, the central angles that intercept them are equal.

P4. The measurement number of a central angle is the same as that of its intercepted arc.

II. Congruent Triangles.

P5. Two triangles are congruent if two sides and the included angle in one are equal respectively to two sides and the included angle in the other.

P6. Two triangles are congruent if one side and the angles next to it in one are equal respectively to one side and the angles next to it in the other.

P7. Two triangles are congruent if the sides of one are equal respectively to the sides of the other.

III. Perpendiculars and Parallels.

P8. Perpendiculars to the same straight line cannot meet.

P9. Two right triangles are congruent if the hypotenuse and an angle next to it, in one, are equal respectively to the hypotenuse and an angle next to it in the other.

P10. If a transversal is perpendicular to one of two parallels it is perpendicular to the other.

P11. In the figure formed by a transversal crossing two straight lines, if those two lines are parallel the alternate angles are equal.

P12. In the figure formed by a transversal crossing two straight lines, if the alternate angles are equal those two lines are parallel.

IV. Parallelograms and the Rectangle.

P13. In any parallelogram the opposite sides are equal, and the opposite angles.

P14. In any quadrilateral, if the opposite sides are equal, or if two sides are equal and also parellel, the figure is a parallelogram.

P15. The area of a rectangle is the product of base and altitude.

V. Measuring Triangles.

P16. The area of a rhomboid is the product of either base and the altitude thereon.

P17. The area of a triangle is the product of any side and the altitude thereon.

P18. Two triangles inscribed in the same stripe have the same ratio as their bases.

VI. Angles and Sides in a Right Triangle.

P19. In every triangle, the angles of a triangle are together equal to 180°; or, the two angles at the base are together equal to the exterior angle at the vertex.

P20. Two right triangles are similar if an acute angle of one is equal to an acute angle of the other.

P21. In any right triangle the square of the hypotenuse is the sum of the squares of the other two sides.

VII. The Law of Cosines.

P22. The cosine of an obtuse angle is the same in value as the cosine of its supplement, but is negative in sign.

P23. In any triangle ABC, $a^2=b^2+c^2—2bc\cos A$.

VIII. The Three Tests of Similarity.

P24. If in two triangles the corresponding sides are proportional, the triangles are similar.

P25. If an angle of one triangle is equal to an angle of another, and the sides including those angles are proportional, the triangles are similar.

P26. If two triangles have two angles of one equal respectively to two angles of another, the triangles are similar.

IX. Ratios of Similar Triangles.

P27. The sine of an obtuse angle is the same as the sine of its supplement.

P28. The area of any triangle is given by the formula $\frac{1}{2}ab\sin C$.

P29. Two similar triangles will have for ratio of areas the square of the ratio of similarity.

X. Similar Figures in General.

P30. In any two similar figures corresponding triangles are all in the same order, or all in reverse order.

P31. Two similar polygons will have for ratio of perimeters the ratio of similarity, and for ratio of areas its square.

P32. Any two similar plane figures will have for ratio of perimeters the ratio of similarity, and for ratio of areas its square.

XI. Measuring the Circle.

P33. The perpendicular to a radius at its point on the circumference is the only tangent at that point.

P34. If a circle is divided into equal arcs, the chords of these arcs form a regular inscribed polygon, and the tangents at the points of division form a regular circumscribed polygon.

P35. If in any circle one regular polygon of n sides be inscribed, and on it another be circumscribed, the two polygons are similar and their ratio of similarity is $\cos\theta$, where θ is $180°/n$.

P36. If a series of regular polygons be inscribed in or circumscribed about a circle, each having half as many sides as the succeeding one, their areas and their perimeters are converging approximations to the area of the circle and to its circumference, respectively.

P37. For every circle the circumference has a constant ratio to the diameter, and the area has the same ratio to the square of the radius.

XII. Loci.

P38. The locus of points equidistant from two given points is the perpendicular bisector of the line joining them.

P39. The angle between a tangent and chord is half the central angle intercepting the same arc.

P40. An inscribed angle is half the central angle intercepting the same arc.

P41. Of triangles all in the same order, having a fixed base and a constant vertical angle, the locus of the vertex is a circular arc.

P42. The locus of the center of a circle inscribed in an angle is the bisector of the angle.

XIII. Concurrent Lines.

P43. Every triangle has one circumcenter only.

P44. Every triangle has one incenter only.

P45. The altitudes from the vertices of a triangle pass all through one point.

P46. In every triangle any medial meets the others two-thirds of the way from vertex to foot.

P47. If r is the radius of the circumcircle of any triangle ABC, then $a = 2r\sin A$, $b = 2r\sin B$, $c = 2r\sin C$. (This is the theorem known as the Law of Sines.)

XIV. INTERNAL AND EXTERNAL SEGMENTS.

P48. In every triangle the bisector of the vertical angle divides the base into segments proportional to the sides adjoining.

P49. In every triangle the bisector of the exterior angle at the vertex divides the base externally into segments proportional to the sides adjoining.

P50. If a fixed point is within a circle, every chord through the point is divided by it into segments whose product is constant.

P51. If a fixed point is outside a circle, every chord in line with the point is divided by it externally into segments whose product is constant and equal to the square of the tangent from the fixed point.

XV. THE NUMBER π.

P52. The sine of half an angle can be obtained by the formula $\sin \theta/2 = \sqrt{\tfrac{1}{2}(1 - \text{Cos}\theta)}$; and the cosine of half an angle can be found from the sine by the formula $\cos^2\theta = 1 - \sin^2\theta$.

P53. The numerical value of π is the limit of $n\sin 180°/n$ as n is successively doubled.

SOLIDS

I. MEASUREMENT OF SOLIDS.

A. *A Study of Prisms*

S1. Planes intersect in straight lines.

S2. If three planes intersect in three lines, these lines either pass through one point or are parallel to each other.

S3. If two intersecting lines in one plane are parallel respectively to two intersecting lines in another plane, the planes are parallel.

S4. If two parallel planes are cut by a transversal plane, the intersections are parallel.

S5. If two lines are each parallel to a third line, they are parallel to each other.

S6. The bases of a prism or cylinder are congruent.

S7. *Problem.* To construct a prism having given a fixed base and a fixed lateral edge.

B. *Volume of a Right Prism*

S8. All the perpendiculars to a given line at a given point lie in one plane which is perpendicular to the line at that point.

S9. *Problem.* To draw through a given point a line perpendicular to a given plane.

S10. Perpendiculars to any given plane are parallel.

S11. *Problem.* To construct a right prism having a given fixed base and the length of a lateral edge.

S12. Two right prisms are congruent if they have congruent bases and equal altitudes.

S13. The measurement number of a right prism on a rectangular base is the product of the measurement numbers of the three edges meeting at any vertex.

S14. The measurement number of any right prism is the product of the measurement numbers of base and altitude.

C. *Volume of a Prism or Cylinder*

S15. The volume of an oblique prism is the product of a right section and the lateral edge.

S16. The volume of any parallelepiped is the product of a base and the altitude on that base.

S17. The volume of any prism is the product of its base and its altitude.

S18. The volume of any cylinder is the product of its base and its altitude.

D. *Volume of a Pyramid or Cone*

S19. The volume of a pyramid or cone has the same measurement number as the area under its curve of sections.

S20. Parallel sections of a pyramid or cone are similar and proportional to the squares of their distances from the vertex.

S21. Pyramids having equal bases and equal altitudes are equal in volume.

S22. The volume of any pyramid is one-third the product of its base and its altitude.

S23. The volume of any cone is one-third the product of its base and its altitude.

S24. The volume of a prismoid having bases B and B', midsection M, and altitude h, is $h/6(B + 4M + B')$.

E. *Volume of a Sphere*

S25. Any plane section of a sphere is a circle.

S26. The volume of a sphere is given by the formula $4\pi r^3/3$.

F. *Areas of Round Surfaces*

S27. The lateral area of a regular pyramid is the product of the slant height and the perimeter of the midsection.

S28. The lateral area of a frustrum of a regular pyramid is the product of the slant height and the perimeter of the midsection.

S29. The lateral area of a cone of revolution is the product of the slant height and the perimeter of the midsection.

S30. The lateral area of a frustrum of a cone of revolution is the product of the slant height and the perimeter of the midsection.

S31. The area of a sphere is given by the formula $4\pi r^2$.

II. Similarity.

S32. Two trihedrals are congruent if their face angles are respectively equal and in the same order.

S33. Two tetrahedrons are similar if the faces of one are similar, respectively, to the faces of the other.

S34. Two similar polyedrons will have for ratio of volumes the cube, and for ratio of areas the square, of the ratio of similarity.

S35. If any two convex solids are similar, they have for ratio of volumes the cube, and for ratio of areas the square, of the ratio of similarity.

III. Loci.

S36. The locus of points equidistant from two given points is the plane perpendicular to the line joining them, at its midpoint.

S37. The locus of points equidistant from the vertices of a triangle is the line through its circumcenter perpendicular to its plane.

S38. The locus of points equidistant from the faces of a diedral is the plane bisecting the diedral.

S39. A sphere can be circumscribed about any tetrahedron.

S40. A sphere can be inscribed in any tetrahedron.

IV. Spherical Figures and Solid Angles,

S41. Spherical triangles are congruent if their sides are respectively equal, and in the same order.

S42. Symmetrical spherical triangles have equal areas.

S43. The area of a lune is $2\ \alpha\ r^2$, where α is the circular measure of the angle of the line.

S44. The area of a spherical triangle is ωr^2, where ω is the number $(\alpha + \beta + \gamma - \pi)$.

S45. The area of a spherical polygon is ωr^2, where ω is the spherical excess of the polygon.

S46. The polar angle of a diedral is the supplement of its plane angle; the polar arc of a spherical angle is the supplement of that angle.

S47. Of two spherical polygons, if one is the polar polygon of the other, then the second is also the polar polygon of the first.

S48. The sum of the sides of a convex spherical polygon is less than 2π.

S49. *Problem.* To find the spherical excess of the solid angle contained by a cone of revolution.

S50. The sum of any two sides of a spherical triangle is greater than the third side.

S51. The shortest distance between two points on the surface of a sphere is the arc of a great circle joining these two points.

FIFTH YEARBOOK

Recreations for the Mathematics Club

By Byron Bentley
Sequoia Union High School, Redwood City, Calif.
Drawings by George Kirchner

Now that the subject of extraclass activities is receiving prominent mention in the field of secondary education, the Mathematics Club is assuming a position of larger importance in the affairs of the school. As with any such organization, however, its value will depend chiefly upon the effectiveness of its programs. Outside speakers will at times be unavailable, business will drag, and even student orators have been known to neglect to prepare themselves upon such weighty topics as "The Life of Pythagoras" or "The Einstein Theory." To enliven the meetings as well as make for constructive accomplishment the recreation has its place.

Nor should the puzzle be regarded as an emergency valve alone. In four out of the five schools whose mathematics club programs are outlined in Chapter XVIII of Roberts[1] and Draper's *Extraclass and Intramural Activities,* one puzzle or fallacy is presented at every meeting, while in the fifth such a presentation might well come under the head of "varied numbers of mathematical interest." Even in the classroom two minutes of this type of play at the beginning of a period will often provide the very stimulus needed in focusing the attention of those more wayward and not infrequently offer an attractive means of approach to some more difficult branch of the subject. It is to furnish material of this kind that the following recreations from the fields of geometry and trigonometry are submitted. Certain old chestnuts are included, but it is hoped that every teacher will find something new to add to his collection. Similar problems may be found in such books as Ball's "Mathematical Recreations" or Dudeney's "Amusements in Mathematics," as well as in the current magazines and newspapers.

Geometry

1. Any point on a line is the midpoint of the line.

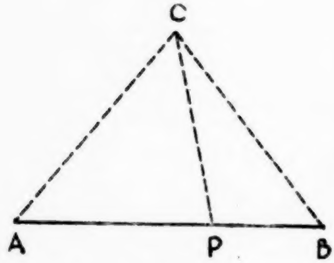


Figure 1

GIVEN: P any point on AB.

TO PROVE: P is the midpoint of AB.

PROOF: ON AB as base construct isosceles triangle ACB and draw CP. Since $AC = BC$, $CP = CP$, and angle $A =$ angle B, triangles ACP and BCP are congruent and $AP = PB$, corresponding parts.

2. Problem 1 suggests the following: Two triangles are congruent when any two sides and an angle of one are equal to the corresponding parts of the other.

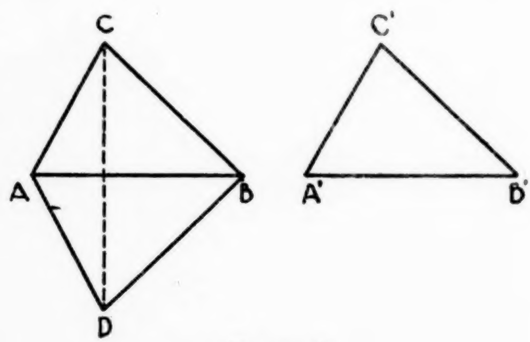


Figure 2

GIVEN: Triangles ABC and $A'B'C'$ with $AB = A'B'$, $BC = B'C'$, and angle $C =$ angle C'.

TO PROVE: Triangle ABC congruent to triangle $A'B'C'$.

PROOF: Place triangle $A'B'C'$ in the position of ABD and draw CD. Since $CB = BD$, angle $DCB =$ angle BDC. Angle $C =$ angle C' by hypothesis. Therefore, by subtraction, angle $ACD =$ angle ADC and $AC = AD$, whence the triangles are congruent having three sides of one equal to three sides of the other.

3. The following was propounded by a Stanford professor to his class. Suppose that a band of iron encircling the earth at the Equator be cut and a strip 20 feet long be inserted. Would the band, if raised uniformly, be high enough to allow the insertion between the

iron and the earth of (1) a piece of tissue paper, (2) a man's hand, (3) a man's body? At the beginning of the year the vote was for tissue paper, at midyears for the hand, and finally for the body. The band would be raised about 3.181 feet in accordance with the following.

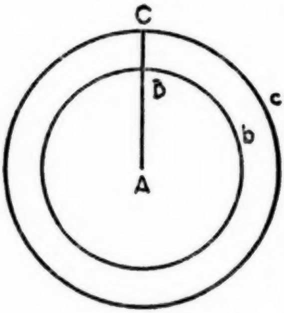


Figure 3

Let g be the circumference of a circle whose radius is BC. Then, since the circumferences of two circles are to each others as their radii, $AC:AB::c:b$, or by division, $BC:AB::c\text{-}b:b$, and by alternation $BC:c\text{-}b::AB:b$. But $BC:g::AB:b$.

Therefore, $BC:c\text{-}b::BC:g$, and $c\text{-}b = g$ or $BC = 20/2\pi = 3.18$ ft.

A. On the subject of circles the next two are in point: To prove that the semicircumference of a circle equals its diameter.

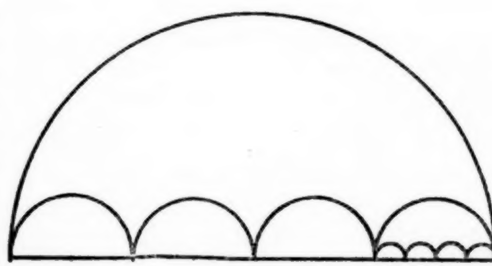


Figure 4

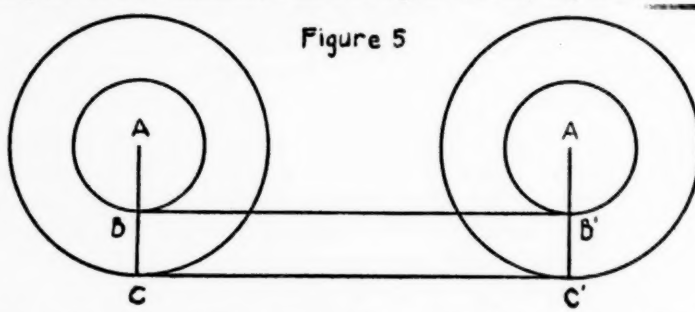
Divide the diameter into 4 equal parts and on each describe a semicircle. The sum of the four semicircles equals the given semicircumference, for the semicircumference equals "pi" $d/2$, and as each small diameter equals $d/4$, the sum of the semicircles equals "pi" $d/2$. Now let each fourth part of the diameter be divided into four equal parts with semicircles described on each. Again the sum equals the given semicircumference. Continue indefinitely until the semicircles coincide with the diameter. Hence, the semicircumference of a circle equals its diameter.

5. To prove that the circumferences of all circles are equal.

Fasten together the centers of two circles of unequal radii as per diagram and let them roll from A to A′. Since the distances BB′

and CC′ are equal and the number of revolutions is the same for both, the two circumferences must be equal. If not, why not?

Figure 5



6. The following are interesting in connection with angles and perpendiculars. To prove that an obtuse angle is equal to a right angle.

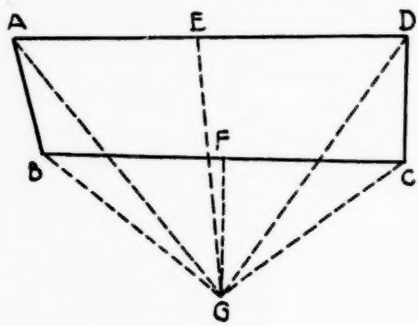


Figure 6

GIVEN: Obtuse angle ABC and right angle DCB.

TO PROVE: Angle ABC=angle DCB.

PROOF: Take AB=DC, draw AD, erect EG and FG perpendicular bisectors of AD and BC respectively letting them meet at G. Draw BG, AG, DG, and CG. Since GA=GD, GB=GC, and AB=DC, triangles ABC and DCG are congruent and angle ABG=angle DCG. But in the isosceles triangle BCG, angle CBG=angle BCG. Hence, by subtraction, angle ABC=angle DCB.

7. To prove that more than one perpendicular may be drawn from a point to a line.

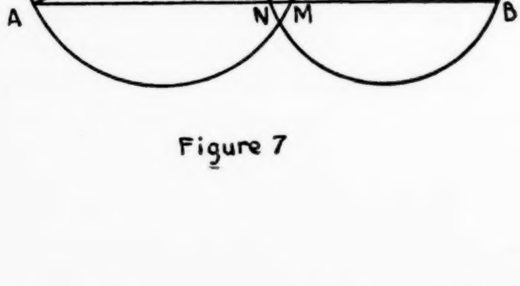


Figure 7

GIVEN: AB and point P outside AB.

REQUIRED: To draw 2 perpendiculars from P to AB.

SOLUTION: Draw AP and BP and construct circles on them as diameters cutting AB at M and N respectively. Then PM and PN will each be perpendicular to AB, since they form right angles inscribed in the semicircles.

8. Another proposition based upon faulty construction is the well-known fallacy that all triangles are isosceles.

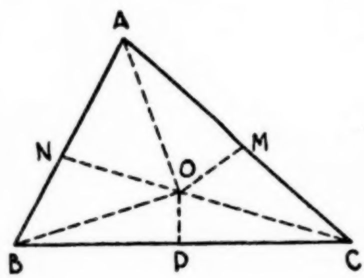


Figure 8

GIVEN: Triangle ABC with AC greater than AB.

TO PROVE: AB=AC.

PROOF: Let OP be the perpendicular bisector of BC and AO the bisector of angle A. Since AC is greater than AB, these lines cannot be parallel. Call their intersection O. Draw OM and ON perpendicular to AC and AB respectively and draw OB and OC. Then in the congruent right triangles AON and AOM, AN=AM. Similarly in ⚠ BON and COM, NB = MC. Hence, AB=AC.

9. The visual deceptions offer a bit of entertainment and may be useful as introductory material in the classroom as well.

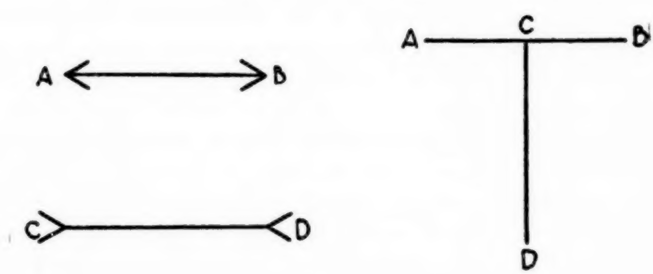


Figure 9

Which is longer, AB or CD?

Are AB and CD in the same straight line in both of the diagrams in Fig. 10 on the following page?

10. One boy brought in a strip of paper one-half inch by ten inches and asked how many surfaces it had. The reply was two, whereupon he pasted the ends together first twisting the strip once, and proved that it had but one surface, since a pencil could be traced all over it without being lifted once.

11. Students beginning the study of areas are often interested in the following: To prove that a square 8 by 8 may be transformed into a rectangle 13 by 5. The same may be cut into sections and fitted into a rectangle as shown in the diagram. Locate the error.

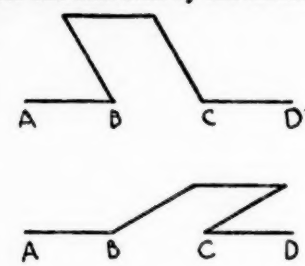


Figure 10

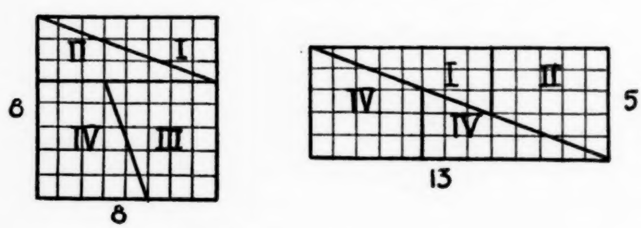


Figure 11

12. Some are unable to apply the Pythagorean theorem to this simple case: A tree 100 ft. tall standing at the edge of a stream broke at the middle so that its top just reached the opposite bank. How wide is the stream?

13. A farmer in cutting a strip around a rectangular field to divide its area in halves, found the width of the strip by taking the sum of the width and length of the field minus one diagonal all divided by 4. Prove geometrically whether or not he was correct.

14. For a construction problem certain people have supposed that the following required the application of the calculus: The bases of an isosceles trapezoid are 9 and 18 inches respectively and the sides are 36 inches each. Draw a line parallel to the bases so as to divide the area of the figure in halves. (If the sides are produced to form a triangle, the upper base cuts off a triangle equal to one fourth the larger, whence the required line will cut off a triangle

equal to five eighths of the whole. Calling one side of this triangle x, and the length of a side of the whole triangle (72″) *a*, $x^2:a^2::5:8$, from which equation x may be constructed.)

15. Perhaps the following should be reserved for the section on trigonometry: A fast half-back is free 40 yards from the side-line with one opponent directly ahead, whom he can outrun by 4ft. to 3ft. Can he escape? If not, how should he lay his course so as to gain the most ground?

16. A few problems in the nature of recreations include the following:

a. Six holes 5/16″ in diameter are to be bored in a piece of wood an equal distance apart in a straight line. The outermost edges of the two end holes are one foot apart. What is the distance between the centers of two consecutive holes?

Ans. 2.3375 inches.

b. A cow is to be tethered at one corner of a one acre field in the shape of an equilateral triangle. How long must the rope be if she is to be allowed to graze over one-half the area of the field?

c. A farmer has his house and barn on the same side of a river. What is the shortest path which will go from the house to the river and then to the barn?

d. Another farmer has his house on one side of a river and his barn on the other a short distance below. He wants to build a bridge at right angles to the bank so that the total distance from house to barn will be as short as possible. Draw the plan and prove it.

e. How many quarters may be fitted around a given quarter, all tangent to it?

f. Two belt wheels 3′8″ and 1′2″ in diameter have their centers 9′5″ apart. Find the length of the belt if it does or does not cross.

g. How much must each of three men grind off a grindstone twenty inches in diameter, if each is to wear off as much as the others?

Solid Geometry

If the club contains members who are interested in Solid Geometry, a report on one of the following problems would furnish instructive entertainment at one of the meetings:

1. Would a round peck measure filled with large onions weigh more than the same measure filled with onions of a much smaller size?

2. If cannon balls one foot in diameter are piled four deep so that each one rests on the four below it, what is the height of the pile?

3. If a six inch sphere is tangent to the floor and to two sides of a room, what is the diameter of the sphere that would be tangent to the given sphere and also to the walls and floor?

4. A two inch hole is bored in a two inch pipe. What is the volume of iron removed?

Trigonometry

For those students who have mastered the fundamentals of trigonometry the following are suggested in addition to the practical problems which may be obtained from almost any engineering corporation in the neighborhood:

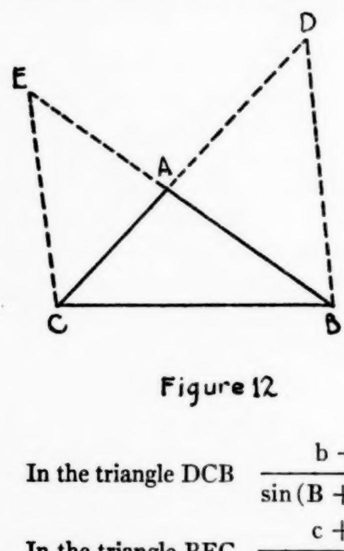


Figure 12

1. To prove that all triangles are equiangular.

GIVEN: Any triangle ABC.

TO PROVE: Angle A = angle B = angle C.

PROOF: Continue AC to D making AD = AB and AB to E making AE = AC. Draw DB and EC. Then angles D and E and DBA and ACE are all equal to one-half angle A.

In the triangle DCB $\dfrac{b+c}{\sin(B+1/2A)} = \dfrac{a}{\sin 1/2A}$ (Law of Sines)

In the triangle BEC $\dfrac{c+b}{\sin(C+1/2A)} = \dfrac{a}{\sin 1/2A}$

Hence, $\dfrac{b+c}{\sin(B+1/2A)} = \dfrac{b+c}{\sin(C+1/2A)}$ and angle B = angle C.

Similarly it may be shown that angle A = angle B = angle C.

2. The average student will perhaps be troubled over the derivation of the formula,

$$\sin A + \sin B = 2 \sin \tfrac{1}{2}(A+B) \cos \tfrac{1}{2}(A-B),$$

if it is pointed out that a generalization is drawn from equations which are true of specific angles only. The derivation is as follows:

Let $A=x+y$ and $B=x-y$ (a specific angle, once x and y are determined). Solving simultaneously, $x=\frac{1}{2}(A+B)$ and $y=\frac{1}{2}(A-B)$.

Since $\sin(x+y) = \sin x \cos y + \cos x \sin y$, and

$$\sin(x-y) = \sin x \cos y - \cos x \sin y, \text{ by addition}$$

$$\sin(x+y) + \sin(x-y) = 2 \sin x \cos y, \text{ or}$$

$$\sin A + \sin B = 2 \sin \tfrac{1}{2}(A+B) \cos \tfrac{1}{2}(A-B).$$

Now, if A contains 80°, x and y may be taken as of any size to add up to that sum, say 60° and 20° respectively, but B can be only one angle, their difference, in this case 40°.

3. A good problem for a report is one after this type: Find the diameter of the earth, if from the top of a mountain three miles high the angle of depression of the horizon is $2°13'50''$.

4. The following is taken from a current magazine: A tree on the side of a hill with a rise of 11′ to 61′ on the slope breaks so that the top may strike the ground 61′ up from the base or 48 and 1/61 feet down from the base. Find the height of the tree.

Ans. 101 feet.

NOTICE TO NATIONAL COUNCIL MEMBERS

We now have 5,600 members in the National Council of Teachers of Mathematics. This is a large increase over the membership we had two years ago, but it is not nearly what it should be. Many teachers are not supporting us at all. One wrote that since his school library was now making the TEACHER, he wished to cancel his subscription. What would happen if we all took this attitude? Let us all talk about the Council everywhere we go, and if you know of a meeting of mathematics to be held any time this winter or spring, write for advertising material to pass out at such meetings indicating how many pamphlets you need. At some of our recent large meetings no mention of the work of the Council was made. This is unfortunate to say the least. Should not some reference be made to the MATHEMATICS TEACHER and to the Yearbooks at every meeting? Miss Constable of the Philadelphia Mathematics Group has sent in over 100 names this year.—THE EDITOR

Whither Algebra?—A Challenge and a Plea

By WILLIAM BETZ

Specialist in Mathematics for the Public Schools of Rochester, New York

INTRODUCTION

The Present Status of Algebra in the High School

FOUR MILLION boys and girls are now enrolled in the high schools of the United States. These young Americans are housed in approximately 25,000 public and 2,800 private high schools. They are being taught by at least 200,000 teachers. These impressive figures alone are perhaps sufficient to explain why the old machinery of education is breaking down. Of necessity, outworn practices are being discarded. Everywhere there is an atmosphere of expectancy, of change and suspense. "What next in secondary education?" has become a slogan that reaches into every nook and corner of the educational edifice. The traditional curricula are being revamped. New objectives are being set up and methods of teaching are being readjusted.

Secondary mathematics, and particularly elementary algebra, can no longer escape this spirit of anxious inquiry, of uncertainty and transformation. In fact, as will be shown below, there probably has never been a time of greater doubt as to its educational significance. Its content, methods, and alleged values are being attacked.

The number of pupils studying algebra in all the public and private high schools in 1928 was more than 1,500,000.[1] Moreover, the great majority of these students were enrolled in the *ninth year of the four-year high schools*.[2] It is claimed that not more than eight per cent of these will actually enter college.

* Reprints of this article may be obtained from *The Mathematics Teacher*, 525 W. 120th St., New York City for 15¢ each prepaid.

[1] According to figures published recently by the United States Bureau of Education, the number of pupils who studied algebra in 16,941 public and private high schools, in the year 1928, was 1,133,930, or 36.1 per cent of the total enrollment of these schools.

[2] "The three-year junior schools numbered 1,053 in 1928, and the two, three, and four-year junior schools numbered 1,403. The six-year high schools, and they include six-year undivided as well as 2-4 and 3-3 schools, numbered 2,237, while the five-year schools numbered 192."

To complete the picture, it is necessary to remember that the high school is enrolling an ever-increasing number of boys and girls from the less favored classes of society. As one principal recently expressed it, "Everybody goes to high school. The genius, the normal, the halt, the lame, and the blind *come to us together and on equal terms.*" This has involved a lowering of "intelligence," if we mean by intelligence the ability to meet traditional standards within the customary time limit of one year. According to careful estimates, at least 200,000 pupils studying elementary algebra each year cannot master the prescribed course within the allotted period of time.

Hence the persistent query: *Is the educational value of elementary algebra sufficiently great to warrant its mandatory inclusion in the daily diet of more than one million boys and girls, many of whom do not appear to have the slightest use for its traditional content or its methods after leaving the high school?*

There are those who wish to solve the entire problem by *differentiated high schools* or by some form of *ability grouping*. Neither of these plans promises to carry us very far. Moreover, neither constitutes a real answer to our question, for the average size of 18,166 public high schools reporting to the U. S. Bureau of Education in 1928 was only 233 pupils. In all but very large communities differentiation of buildings and of courses is out of the question. Besides, if *differentiation of schools* were *theoretically* possible, it is a grave mistake to urge early specialization or vocationalism, since no one can guarantee a satisfactory life career resting on so narrow or insecure a background. As to *ability grouping*, not a single reliable experiment has been reported that warrants wholehearted imitation.[3] On the contrary, well-nigh insuperable administrative difficulties have all but defeated the plan even in large schools that have given it an impartial trial.[4] The voluminous report presented in the Twenty-Fourth Yearbook of the National Society for the Study of Education (1925) proves that the problem is one of staggering difficulty.

Finally, the attempt to *convert ninth-year mathematics in the four-year high school into a semblance of general mathematics,* after the fashion of the best junior high school curricula, raises serious problems.

[3] Symonds, Percival M. *Measurement in Secondary Education,* p. 478. The Macmillan Company, 1928.

[4] Mort, Paul R. The Individual Pupil, p. 334. American Book Company, 1928.

What does "general mathematics" mean, *in such a case?* As a rule, such a course becomes a mixture of business arithmetic, with a return to the badly overworked percentage game, of "practical" mensuration, incorrectly called intuitive geometry, and of a largely imitative manipulation of formulas and equations. That is, the plan virtually eliminates algebra. Whatever one may think of the merits of such a procedure, this type of "general mathematics" does not assist us in solving our main problem, that of establishing the teaching of algebra on a more secure foundation.[5]

It is not an accident, therefore, that all competent critics now unite in regarding the ninth year as the storm center of the progressive educational movement. In the following pages the attempt will be made to analyze some of the causes of the prevailing unrest with reference to elementary algebra, and to suggest a reorientation that seems to be an unavoidable outcome of the present situation. The views presented will not be those of an alarmist nor of one who habitually indulges in negative criticism, but those of an experienced teacher who still believes most emphatically in the permanent educational importance of secondary mathematics.

PART ONE

The Challenge

1. *The Verdict of the General Educator.* If you had never heard of algebra previously, and were obliged to derive your opinion of the subject from its rating at the hands of the general educator, your impression would be far from reassuring. A few characteristic quotations will serve to prove this statement.

One of the most radical opponents of the present high school organization is Professor D. Snedden of Teachers College, Columbia University. In a recent lecture outline entitled "Some Forecasts of Curriculum Changes in High Schools," Professor Snedden unburdens his soul of the following complimentary utterances:

> Many of our older high school offerings—algebra, French, ancient history, chemistry, classical literature—however educationally nutritious they may be for upper quarter intelligences, are educational *sawdust* and *gravel* for second quarter intelligences. Hence we must find high schools or high school departments or curricula or at least courses for near-morons—whose educational souls we must

[5] A comprehensive study of this whole question is now available. See McCormack, Clarence, ***The Teaching of General Mathematics in the Secondary Schools of the United States.*** Teachers College Bureau of Publications, 1929.

help to save just as certainly as those of the brainy ones. But plane geometry, Latin, Mediaeval history, or advanced composition will *not* save the educational souls of those less than most able. Neither will commercial studies or shop studies or agricultural studies or any other largely sham vocational offerings suffice, though these may be more assimilable than the sawdust French or algebra. —*When we get rid of present superstitions we shall certainly advise not over ten per cent of our students to take algebra (for strictly prevocational reasons) and we shall actually prevent at least fifty per cent from taking it, as we should prevent most healthy people from taking quinine or morphine.*[6]

Professor Bobbitt's views are too well known to require extensive restatement. After branding the objectives suggested by the National Committee on Mathematical Requirements as "purely academic," he considers the "real" function of secondary mathematics. In his monograph on "Curriculum-Making in Los Angeles" (1922), he writes,

Algebra, geometry, and trigonometry have a justifiable place in the curriculum *only when they are necessary portions of vocational courses* . . . . The major thing needed is not ability to solve difficult mathematical problems; it is rather ability and disposition to think accurately and quantitatively in one's affairs. The latter frequently involves *mathematical operations as incidental matters—never as the fundamental ones.*—Outside of their vocations, the citizens of Los Angeles do not use algebra, demonstrational geometry, or trigonometry. . . . . Outside of their vocations, *the only mathematics content really needed by the men and women of the city is applied arithmetic.* . . . . Even in their vocations, only a small percentage of the citizens of Los Angeles use algebra or trigonometry; and practically none use demonstrational geometry. . . . . The mathematics needed for one's vocation should be determined strictly with a view to that vocation. It should then be administered only to those who enter that vocation; and it should be very thorough, especially along the applied lines involved in that vocation. . . . . *As fields of intellectual play, neither algebra nor demonstrational geometry lay foundations or centers of systems of ideas and thought generally needed throughout life.* . . . . As matters of pure general discipline the city cannot afford to administer algebra and geometry purely on faith; the specific disciplinary values should be made clear; and *it should be demonstrated that they are or can be attained.* . . . . The value of *applied* mathematics, intensive and thorough, for producing power to think, to assemble and organize facts, and so on, has been amply demonstrated. . . . . *The needed mastery of the world of number is to be attained mainly through using number—not by studying abstractions about number.*[7]

In a passage that was recently given wide publicity by its conspicuous appearance in a research bulletin issued by the National Education

[6] The italics are mine in this and in all other quotations used in this discussion.

[7] For a more detailed presentation of Professor Bobbitt's views, see *The Mathematics Teacher,* Vol. XVI, pp. 455-459.

Association,[8] Professor George S. Counts pays his respects to secondary mathematics in the following manner:

> A word should be said regarding the *reluctance of the high school to abandon algebra and geometry.* The case against these subjects as to the basic offerings in mathematics for the great majority of high school pupils is clear. *They reached the place which they occupy in secondary education largely by historical accident and by the grace of the doctrine of formal discipline. They contain but little material that is related to either the present or the probable needs of the pupils.* Experimentation indicates that even for that small group of individuals who possess great mathematical talent and who should be encouraged to pursue higher mathematics, some other organization of mathematics would give superior training; yet, *the composite mathematics courses have made but little headway in these schools.* . . . . Undoubtedly, the intrenched position which algebra and geometry hold in the college entrance requirements have much to do with their persistence in the high school curriculum.[9]

This list of painful quotations could easily be extended. There may be those who can dismiss such unpleasant appraisals with a gesture of sarcastic resentment or of tolerant amusement. Such an attitude of indifference is hardly possible, however, when one of the sanest and most influential educational thinkers in America, Professor Charles H. Judd of the University of Chicago, finds is necessary to be equally severe in his condemnation of the prevailing type of mathematical teaching. In his stimulating address before the National Council of Teachers of Mathematics, at the Cleveland meeting in February, 1929, he went so far as to predict the temporary elimination of mathematics from the curriculum, pending a complete change in its customary content and its methods.[10]

Any intelligent layman reading these and similar opinions would naturally infer that a subject which can merit such unlimited criticism is in urgent need of repair, and that it must either take stock of its resources or face the imminent prospect of bankruptcy.

2. *The Verdict of Examination Results.* Mathematics is universally rated as a "high mortality" subject. This may be an unavoid-

[8] Research Bulletin, entitled *"Vitalizing the High School Curriculum."* Washington, 1929.

[9] Counts, George S., *The Senior High School Curriculum.* The University of Chicago Press, 1926.

[10] Reprints of Professor Judd's valuable address, on "Informational Mathematics versus Computational Mathematics," may be obtained from the office of *The Mathematics Teacher*, 525 West 120th Street, New York City, for 5 cents per copy.

able consequence of its logical and cumulative character. Nevertheless, this fact does not enhance its popularity or its educational security.

According to the results of a recent survey of the New York City high schools, in which about 2,000,000 ratings were tabulated, "mathematics causes the greatest number of casualties among high school students." Twenty-two or twenty-three out of every hundred fail in this subject. The mortality in algebra alone appears to be much higher. In fact, several years ago the average per cent of pupils passing Regents examinations in elementary algebra in twenty high schools of New York City was 57 per cent.

The following tables, based on statistics published by the Regents of the State of New York, and by the College Entrance Examination Board, will furnish interesting material for reflection.

TABLE I. RESULTS OF REGENTS EXAMINATIONS IN ALGEBRA AND GEOMETRY

Subject	1927		1928		1929	
	a	b	a	b	a	b
Elementary Algebra	86,276	70.7%	96,928	73.9%	97,178	73.7%
Intermediate Algebra	31,375	81.0	34,605	77.7	36,750	59.8
Plane Geometry	58,512	72.5	60,089	77.1	64,220	68.3
All mathematical subjects	188,915	73.7	206,773	76.4	215,413	70.1

Column *a* indicates the number of papers written, while column *b* gives the per cent of papers accepted (65%–100%).

The pupils taking the examinations of the College Entrance Examination Board are supposed to be a highly selected group. And yet, consider the following results, column *b* indicating the per cent of papers rated 60%-100%:

TABLE II. RESULTS OF COLLEGE ENTRANCE BOARD EXAMINATIONS—(OLD PLAN)

Subject	1927		1928		1929	
	a	b	a	b	a	b
Elementary Algebra	7,058	64.6%	7,171	67.4%	6,625	68.6%
Algebra to Quadratics	765	37.4	700	42.9	606	65.3
Quadratics and Beyond	291	56.7	217	55.4	218	56.9
Plane Geometry	6,179	67.4	6,201	72.5	5,912	74.7
All mathematical subjects	18,918	64.0	18,863	69.4	17,563	69.6

More damaging still are Professor Thorndike's investigations as reported in his *Psychology of Algebra*. A test consisting of forty rather

simple algebraic exercises was given in ten schools. "These schools were either private schools with excellent facilities, or public high schools in cities which rank *much above the average of the country* in their provision for education. *All the pupils had studied algebra for at least one year.*" The summary of results showed that *"these students of algebra had mastery of nothing whatsoever. There was literally nothing in the test that they could do with anything like 100% efficiency."* In the case of some of the simplest questions the percentage of correct responses in one of the schools was as low as 5 per cent.[11]

Professor Thorndike is of the opinion that a pupil whose intelligence quotient is below 110 "will be unable to understand the symbolism, generalizations, and proofs of algebra. He may pass the course, but he will not really have learned algebra. This would rule out more than half (56 per cent) of the present first-year students."[12]

It may be somewhat consoling that similar examination results are in evidence in practically all other high school subjects. Hence the general educator at present is attacking *all* high school subjects *as now organized,* chief attention being given to "the greatest sinners," algebra and Latin. In endorsing the wide acceptance of "the new philosophy of education," Professor S. A. Courtis recently commented as follows:

> "The old education proved to be so ineffective that any new method couldn't be worse. On every count our schools have been proved to be grossly and hopelessly inefficient. Better ways of teaching became imperative."[13]

3. *The Verdict of Educational Theory.* The pragmatic philosophy of education made popular by Dewey and his followers has all but gained complete ascendancy in our elementary schools. Its fundamental postulates have become part of the established creed of the primary teacher. A program of "activities" or "projects" is rapidly replacing or supplementing the stereotyped elementary curriculum. Education is defined as growth, as "reconstruction of experience" by purposeful self-activity. In these "child-centered schools" there is a horror of all "traditional subject-matter," of formalism and "imposed" goals. Instead of a mere preparation *for* life, "education *is* life."

[11] Thorndike, E. L. *The Psychology of Algebra,* p. 321. The Macmillan Co., 1923.

[12] Ibid., p. 37.

[13] See *Educational Method,* Vol. IX, November, 1929.

The doctrine of formal discipline has all but vanished. Its successor, the theory of the transfer of training, insists that worthwhile transfer can come only through ***meanings,*** through ***flexible concepts*** and ***significant applications.***[14]

This "new education" is now knocking at the gates of the high school. Its demands can no longer be ignored. How completely ordinary algebra teaching is at variance with the prevailing type of educational theory will be shown below. This discrepancy has served to increase the ill repute of secondary mathematics.

About twenty years ago William McAndrew, formerly Superintendent of Schools of Chicago, in an address before the New Jersey High School Teachers' Association, characterized the backwardness of secondary mathematics in a manner that would still meet with the approval of the average modern "curriculum expert." He said,

> A peculiar kind of mathematics is a staple of high school procedure, algebra and geometry: recitations every day for two years with possibly some more for the remnant of the students, principally girls, who resist the expulsive processes of the first two years.
>
> I have taught and supervised these subjects for twenty-two years. It may not be especially edifying to you to have me ***confess my own inability to see any correspondence between high school mathematics and the educative processes of real life.*** It may be there, but do the mathematics teachers parade it offensively before the view?
>
> Yes, I know, when we are cornered we can twist an argument to show that the mental power that comes from these exercises is very valuable in many of the problems of life, but ***how much reference to such problems do you hear in high school mathematics classes?***[15]

4. *The Verdict of the National Committee on Mathematical Requirements.* For nearly thirty years we have had such a thing as a "reform movement in secondary mathematics."[16] It found its culmination when, in 1916, the National Committee on Mathematical Requirements was organized under the auspices of the Mathematical Association of America. This Committee consisted of six outstanding representatives of the colleges and universities, including such men as Professor E. H. Moore of the University of Chicago, and of seven

[14] See "The Transfer of Training," by William Betz, in the *Fifth Yearbook of the National Council of Teachers of Mathematics.*

[15] Quoted from a printed report issued by the Association.

[16] A splendid review of this movement was submitted by Professor D. E. Smith in the *First Yearbook of the National Council of Teachers of Mathematics.*

representatives from the field of secondary mathematics. After several years of careful and extensive preliminary work, the Committee, in 1923, issued its famous Report on *The Reorganization of Mathematics in Secondary Education*. This document was widely distributed and was discussed in countless meetings. Its message was actively carried forward by THE MATHEMATICS TEACHER.

The National Committee framed a very sensible new syllabus in elementary algebra, insisting throughout on simplicity and coherence. It offered an excellent set of suggestions as to the "point of view" which should govern instruction. It emphasized *insight, appreciation,* and *effective habits of thinking.* It warned against *"isolated and unrelated details."* Instead, it recommended that the entire course be dominated by "certain *general* ideas." Again and again the National Committee called attention to the danger of mechanization. It asserted that *"the excessive emphasis now commonly placed on manipulation is one of the main obstacles to intelligent progress. Drill in algebraic manipulation must be conceived throughout as a means to an end, not as an end in itself."* Above all, *a program of functional thinking was suggested as the unifying element of the entire course.*

Without question the "National Report" has been the most powerful single factor in the transformation of secondary mathematics. To thousands of teachers it brought a new vision, and the leaven which it contains will be an active antidote against stagnation and reaction.

In spite of all that was said above, there is ample evidence that even the splendid effort of the National Committee has failed thus far to modify the prevailing classroom procedures to a considerable extent. The imitative juggling of symbols continues as of old. Problems are still considered an unpleasant incident. The whole course is still an aimless array of "isolated and unrelated details." Textbooks are still advertised and rated largely on the basis of the number of *drill* exercises they contain. Obsolete and useless material is still perpetuated in abundance. Examinations are only beginning to react to the suggestions of the National Committee. Functional thinking is a sort of homeless ghost that is honored by an occasional oblation in the form of a respectful foot-note or a carefully detached appendix.

In short, we have sung many hymns of praise in honor of the National Report. But our actual schoolroom and examination routine proves that we have indulged merely in lip service.

PART TWO

Outstanding Defects of Current Instruction in Algebra

1. *Absence of a Central Theme.* It should no longer be regarded as a species of original sin on the part of an inquisitive pupil if he asks his algebra teacher, on the very first day, *"What is it all about?"* or *"What is the use of studying algebra?"*[17] As a rule, such a pupil would be unable, even with the most powerful microscope, to find a satisfactory answer to these questions in his textbook. And yet, the National Committee clearly suggested a central theme for a modern course in algebra.

Why all this backwardness? The answer is twofold. We still regard unrelated *technique,* and not *thinking,* as the essential part of algebra teaching. Our textbooks and examinations still continue to reflect that deplorable error. The individual teacher feels that, since she is held responsible primarily for her examination results, she must follow the beaten path. And so the "vicious circle" continues.[18] It can be broken only by the realization that the customary plan is no longer desired by either progressive colleges or examining bodies, that it serves no useful purpose, and that it defeats the primary aim of mathematical teaching. By *insisting* on the modern point of view, individually and collectively, algebra teachers will soon be able to effect a readjustment in textbooks and examinations, as have the teachers of English, of Latin, of Social Science, and of other high school subjects.

2. *The Curse of "Empty Symbolism."* It is hard to discuss the futility of mere symbolism without being misunderstood. For algebra is a kind of shorthand which "says more, in fewer words, than any other language." The symbolism of algebra is its glory. But it is also its curse.

The invention of symbols, according to Professor Dewey, was the greatest single event in human history.[19] It made possible the detachment of thinking from a concrete environment. It is the basis of all abstract thinking. Now, the development of symbolism, as every one

[17] Cf. "A Layman Looks at Mathematics," by John R. P. French, *The Mathematics Teacher,* October, 1929, p. 352.

[18] See *National Report,* pp. 51-54.

[19] Dewey, John, *The Quest for Certainty,* p. 151. Minton, Balch and Company, 1929.

knows, was very slow.[20] Its essential presupposition was the creation of *concepts*. The gigantic task of formulating a conceptual world undoubtedly has been going on since the first dawn of human consciousness.

But what is a *concept?* Modern psychology regards a concept as a *"substitute for a previous experience."*[21] For example, instead of being obliged to deal exclusively with concrete objects, I can deal *mentally* with the corresponding "concepts." This gives me the *power of regarding objects as present realities even though they may be absent.* Thus, by using the *concept* "rectangle," I am no longer compelled to limit myself to an *actual object* of rectangular form. I can conjure such an object at any time in a purely *mental* fashion.

A *symbol* arises when I use a word, or a written sign, to *refer to a concept.* Thus, the spoken or written word "tree" serves to recall the concept designated by the symbol. Hence *such a symbol is likewise a "substitute for experience."* This suggests that if the concept for which the symbol is supposed to stand does not correspond to anything that has been experienced, the symbol has no meaning. *But a symbol which does not symbolize something is useless.* It is "empty." At best, it may serve as a form of amusement, but *educationally* it is barren.

In like manner, when a pupil uses the technical symbols of algebra, before he understands what they symbolize, or how he can apply them, he is indulging in a blind game that has no intrinsic educational value whatever.

Bertrand Russell's playful definition of mathematics as *"the science in which we never know what we are talking about"*[22] seems to be only too true a description of algebra as it is reflected in the mind of the average pupil. An inspection of the textbooks, even the recent ones, proves that all but a very few quickly plunge into the usual manipulations of meaningless symbols.

A subject which deliberately tolerates or encourages such a game of empty juggling may furnish a speculative stimulus to specialists.

[20] See Judd, Charles H., *The Psychology of Social Institutions*, Chapters IX and X. The MacMillan Company, 1927.

[21] See Bode, Boyd H., *Fundamentals of Education*, pp. 107 ff. The MacMillan Company, 1921.

[22] Quoted in Young's *Fundamental Concepts of Algebra and Geometry*, p. 3. The MacMillan Company, 1911.

But when it parades that sort of thing as mental food for young pupils, it is committing educational suicide.

3. *Imitative, Mechanical Learning.* One educational sin is bound to lead to others. Since empty symbols suggest no ideas to the young learner, the operations of *combining* these symbols become even more mysterious. Hence teachers, in sheer desperation, soon tolerate or encourage a purely imitative performance. For the pupil "*must*" learn these "fundamental" operations. When insight and motive are completely absent, nothing but incessant mechanical drill will work. And so algebra becomes a "muscular" reaction, a type of "physiological learning" or of "animal learning."

Fortunately, we are now beginning to understand the futility and the monstrous misdirection of that type of "teaching." *The fundamental error lies in a wrong psychology of learning.* Abundant proof of this assertion will be found in the recent treatises on the psychology of thinking.

A careful distinction should be made between 1) automatic reflexes, which are unlearned, 2) conditioned or acquired reflexes, and 3) all forms of purposive or self-directed learning. The mechanisms involved in all these types of response have recently been described by Professor C. J. Herrick of the University of Chicago in a book of very great importance. To quote,

> Human learning may be divided into two stages or phases. One of these is the sort which is common to all kinds of living things, common protoplasmic learning. This is going on from the very beginning of the individual and it lasts throughout his lifetime. . . . . The other kind is *intelligently directed learning which is a function of the cerebral cortex.* It begins some time after birth and it will go on more or less efficiently throughout life, depending on the amount and quality of cortex available and the opportunities for its exercise provided by the education of the child. Both of these kinds of learning are done more rapidly and more easily in childhood than in old age, though intelligently directed learning in some people does not attain its maximum efficiency until middle life. . . . . These two kinds of learning we call for short *physiological learning* and *intelligently directed* or *purposive learning.* . . . . *Physiological learning is a far more primitive sort of thing than purposive learning,* and the latter has evidently grown up within and out of the more primitive sort.[38]

In a special chapter, Professor Herrick describes the importance of "intuition and insight." After stating that "*learning by rote, learning*

[38] Herrick, C. J., *The Thinking Machine,* pp. 233 ff. The University of Chicago Press, 1929.

by trial-and-error, and *learning by conditioning of reflexes are apparently* fundamentally the same," he says, "In men and apes and probably in some other higher animals there are kinds of learning that do not fit into this scheme." Among these is what he calls "learning by insight." In such a case, *we "see through" a situation before we make a move.* To quote,

> Give a man (or a dog) just *one* experience that has *meaning,* that fits into a frame of previous experience, and so finds the higher nerve centers already set up for an appropriate reaction, then the *learning process is very different.* The one new experience is *immediately assimilated* to old knowledge and familiar ways of behaving, and *it is remembered perhaps for the rest of his life.* . . . . Learning by insight, whether intuitively or by consciously directed effort, has the same adaptive value that all conscious capacities have. The ability to "see through" a situation without wasteful fumbling adds to competence as well as to *speed of reaction.* It requires a very complicated nervous system and it gives us far better ability to thread our way through the intricacies of unaccustomed and often baffling situations. . . . . When we see otherwise intelligent people fumbling around in situations where *a little good headwork* would take them through without blundering, we wonder whether our schools are really educating people or merely teaching them.[24]

Professor Bode, in his acute analysis of the learning process, presents a scathing arraignment of the customary emphasis on mechanical habit formation in much of our school work. He calls attention to the fallacies of the prevailing stimulus-response type of psychology. In reality, such "animal learning" gives to the pupil merely "a bag of tricks." He characterizes this sort of teaching as "an enemy of democracy."[25]

Instead, by emphasizing *meanings* and *relations,* we could produce an *economy* of learning that few teachers suspect. Professor Herrick stresses the truth that by cultivating *insight* we can often learn in a few *seconds* what might otherwise require wearisome, endless repetition, as in the case of animals.

Above all, "transfer of training" appears to take place only through *meanings* and *concepts.*[26] Hence all learning which does not rest on *insight* and thinking is bound to be *educationally* barren.

When these facts are realized by intelligent teachers, we shall hear

[24] Ibid., pp. 224 ff.

[25] Bode, Boyd H., *Modern Educational Theories,* p. 186. The Macmillan Company, 1929.

[26] Ibid., pp. 202 ff.

less of "empty symbols," and algebra will recover from the drug effect ascribed to it by Professor Snedden.

4. *Dissociation of Technique from Application.* Another basic principle of modern educational theory is the importance of continuous and many-sided *application.* Now a mechanical juggling of meaningless symbols makes real application almost impossible. Hence we have in algebra the customary *dualism of technique and problem-solving.* These two ingredients are kept in separate compartments. They will not mix. When the symbolism and the skills are not acquired in meaningful associations, they cannot suddenly be made to function in a significant way. Textbooks and examinations foster this dualism. Hence the teacher spends about 80 per cent of her time on isolated technique and not more than 20 per cent on problem-solving. It is hard to imagine a greater deviation from the course that *should* be followed.

PART THREE

Importance of a New Orientation

1. *The Central Theme of Algebra.* The National Report rendered a signal service in suggesting an admirably succinct platform for all mathematical instruction. To quote,

The primary purpose of the teaching of mathematics should be

1) to develop those powers of understanding and of analyzing relations of quantity and of space which are necessary to an insight into and control over our environment and to an appreciation of the progress of civilization in its various aspects; and

2) to develop those habits of thought and of action which will make these powers effective in the life of the individual.

For the field of algebra, the National Report formulated the following guiding principles:

The one great idea which is best adapted to unify the course is that of the functional relation. The concept of a variable and of the dependence of one variable upon another is of fundamental importance to everyone. It is true that the general and abstract form of these concepts can become significant to the pupil only as a result of very considerable mathematical experience and training. There is nothing in either concept, however, which prevents the presentation of specific

concrete examples and illustrations of dependence even in the early parts of the course. Means to this end will be found in connection with the tabulation of data and the study of the formula and of the graph and of their uses.

The primary and underlying principle of the course should be the idea of relationship between variables, including the methods of determining and expressing such relationships. The teacher should have this idea constantly in mind, and the pupil's advancement should be consciously directed along the lines which will present first one and then another of the ideas upon which finally the formation of the general concept of functionality depends.[27]

Now, it is the *spirit* of these suggestions, not their mechanical observance, which is of primary importance. As yet, textbooks and teachers still seem to be following the line of least resistance. The usual policy of indulging in a few *isolated* references to these ideas gets us nowhere. Relational thinking must become a *dominant melody* permeating the whole course if it is to bear fruit.

Mr. Breslich, after a thorough analysis of four widely used textbooks, states that "although on almost every page *opportunities* for training in functional thinking are offered, no systematic use of them is made."[28] Professor Hedrick has pointed out, again and again, that functional thinking is not a matter of *symbols* or of definitions. The material for a continuous and fruitful study of dependence and of relationships surrounds us on all sides. Hence it should not be the exclusive task of the *textbook* to provide "opportunities for functional thinking." Actual everyday illustrations and problems suggested by the environment and the interests of the pupils will be vastly superior in their stimulating freshness to stereotyped, printed exercises. Nevertheless, our textbooks should give much more assistance than they do. In the college field, the emphasis on functional thinking has become thoroughly established.[29] As soon as algebra teachers begin to appreciate the gold mine of splendid mathematical training which they can appropriate and make available almost without effort, we may

[27] See *National Report,* Chapter II, p. 12.

[28] See Breslich, E. R. *Developing Functional Thinking in the High School,* in the Third Yearbook of the National Council of Teachers of Mathematics.

[29] Teachers should have constant access to the newer college texts, such as those of Gale and Watkeys; Young and Morgan; F. L. Griffin; Karpinski, Benedict and Calhoun.

expect a veritable revolution in elementary mathematical teaching.[80]

2. *Importance of a Functional Program.* The argument in favor of such a program, even in elementary classes, is very simple. The world in which we live is one of ceaseless change. All these changes have their causes. The study of changing phenomena and of their underlying causes has engaged the attention of man ever since the dawn of human history. Even primitive man was aware of a regularity or rhythm in such phenomena as the recurrence of day and night, of the seasons, and of the lunar phases. At a later stage he observed the orderly procession of the heavenly bodies. The constellations regularly appeared in their accustomed positions and thus served to fix the calendar. By slow degrees these beginnings of *scientific thinking* were extended to every domain of the physical world. *Modern science has proved that there is regularity, or rhythm, in all the phenomena of nature,* from the motion of the giant suns in distant regions of the universe to the movements of the tiniest electron. And *all these phenomena are interrelated. Only by understanding these rhythms and relationships can we hope to deal intelligently with the world in which we live.*

More than that, nature's secrets become a source of *power,* of *control,* only when they can be formulated in a concise, objective manner. This is precisely the rôle and the purpose of the *formula. Algebra has contributed the machinery for expressing in a universal code the cosmic relationships which have become the alphabet of science and industry.* The formulas used by the engineer and the scientist are the cumulative result of age-long investigation and effort. Hence they represent a treasure house of information,—an armory of priceless value.

If the symbolism of algebra can be directed from the beginning, even in a rudimentary way, toward the goal suggested above, *algebra ceases to be a mechanical game and becomes, instead, a means of initiating the young learner into a domain of transcendant interest and permanent value.*

3. *Importance of Central Objectives.* The curriculum revision movement has at last made teachers familiar with the idea that the ineffectiveness of the older curricula was largely due to the absence

[80] For a simple discussion of this question, see the *Tentative Syllabus in Junior High School Mathematics,* issued by the New York State Department of Education, Albany, New York.

of a carefully formulated group of tested and valid objectives. In the case of algebra there is a growing agreement that practically all the worthwhile material of the introductory course can be organized around the following central objectives:

1. *The language and the ideas of algebra.*
2. *The formula.*
3. *The equation.*
4. *The graph.*
5. *The fundamental principles and processes.*
6. *Problem-solving.*

Among these objectives, the formula, the equation, and the graph represent the functional core. They should be regarded as the "Big Three" of the curriculum. If the entire course is based on such a small body of objectives, we shall not only achieve a far greater economy and concentration, but shall also remove the prevailing impression that the customary course is an aimless array of "isolated and irrelevant details."

4. *Algebra as a Language and as a Type of Thinking.* The symbolism of algebra is of educational value only when it expresses ideas that are of permanent importance to boys and girls. This is the case when these symbols are used 1) to express mathematical laws in concise and general form; 2) as tools for the economical statement and mastery of quantitative relationships occurring in nature, science, technology, or everyday life; 3) for the solution of significant quantitative problems. If the processes of algebra are made to contribute to the gradual unfolding of such a program, they serve a useful educational purpose.

It is impossible to realize this ideal without a constant insistence on *understanding,* on real *application,* and on actual *thinking.* As Bode puts it, "The power to think is the educational Kingdom of Heaven; if we seek it persistently, other things will be added unto us."[31] Teachers who have once honestly tried to replace the customary formalism by the joy of genuine thinking, will never want to go back to the slave-driving methods necessitated by mechanical drill.

5. *Mastery of the Essential Skills.* It has been pointed out that *excessive attention to mere manipulation has long been the outstanding weakness of algebraic teaching.* We certainly need a different distribution of emphasis. Much progress would result if we could secure the

[31] Bode, B. H. *Conflicting Psychologies of Learning*, p. 274.

general adoption of the wise counsel given by the National Committee. To quote,

"Drill in algebraic manipulation should be limited to those processes and to the degree of complexity required for a thorough understanding of principles and for probable applications either in common life or in subsequent courses which a substantial proportion of the pupils will take. It must be conceived throughout as a means to an end, not as an end in itself. Within these limits, skill in algebraic manipulation is important, and drill in this subject should be extended far enough to enable the students to carry out the essential processes both accurately and expeditiously."[32]

There can be no question that a mastery of the essential skills is absolutely imperative. No real appreciation of mathematics is possible, and no worthwhile application, when a pupil is constantly handicapped by inefficiency in the elementary processes. Mere "exposure" to such a technique is useless. A high degree of mastery is therefore to be secured in connection with each of the following topics: 1) the four fundamental operations involving ordinary integral algebraic expressions; 2) the solution of simple equations in one or two unknowns; 3) the evaluation of algebraic expressions or formulas; 4) the construction and interpretation of ordinary graphs, both statistical and mathematical; 5) the finding of numerical square roots. In the case of algebraic fractions the emphasis should be placed on monomial or binomial denominators. Factoring should be regarded as an incidental topic. All the important recent syllabi stress but three cases in factoring.

Such a restricted program may at first appear too "emasculated" to a conservative teacher. The truth is, as was shown above, that the customary attempt to habituate *hundreds* of "bonds" in the brief period of one year leads to confusion and general inefficiency. Above all, the core which has been suggested is the only one that can really be applied successfully in an introductory course.

6. *The Psychology of Drill.* No one questions the importance of *purposeful* drill. Efficient application is impossible without adequate tools. The idea that somehow the skills can be acquired *incidentally,* or without effort, can be disproved in any ordinary classroom. But there is a vast difference between blind imitation and purposeful learning. Hence *it is a fundamental error to base the technique of algebra*

[32] See National Report, p. 11.

on unmotivated or unrelated rules. A rule of procedure, in elementary mathematics, should not be merely a "rule o' thumb," but rather *the summary of a rationalized and motivated process.* In general, it should be the *culmination of a preliminary investigation, not the starting point.* It should be based on consciously recognized *principles.* When so conceived, *a rule is an epitomized learning process.*

Hence, whenever a pupil is in doubt about the "next step," he should be trained to think of the underlying *principles,* instead of depending exclusively on memorized *rules.* There are two principal reasons why a merely mechanical and imitative technique is inadequate.

First, the number of algebraic abilities or "bonds" to be habituated in the course of a year is too large to be assimilated by mechanical learning alone.

This is amply proved by the examination results quoted above. If pupils fail to master the simple skills of *arithmetic* in eight years, how can we expect them to master a far more extensive domain, in a few months, by mere imitation?

Second, these skills appear in so many different combinations that it would be a hopeless task to depend entirely on a set of automatic push-button reflexes.

How can we cause these numerous "bonds" to function, either individually or collectively, *in precisely the correct order,* by *mechanical* habituation? Even Professor Thorndike's theory of "conduction units" fails to help us out in such a case.

Thus, if the solution of simple equations rests primarily on a single rule, that of "transposition," the numerous types of equations that the pupil is expected to deal with soon get him "all mixed up."

For example, Professor W. D. Reeve reports the results of a composite test on formal equations, which was taken by 1,204 pupils after thirteen lessons in the subject had been given.[38]

Observe the percentages of correct responses in the following table:

Equation	*Per cent of Correct Responses*
1. $x + 5 = 9$.	97.5
2. $x - 3 = 4$.	93.4
3. $8x = 5x + 12$.	89.3
4. $4x + 5 = 17$.	95.7
5. $y/_3 + 2 = 5$.	64.3
6. $x/_3 - 5 = 9$.	53.7

[38] Reeve, W. D., *A Diagnostic Study of the Teaching Problems in High-School Mathematics,* pp. 33 and 104. Ginn and Company, 1926.

How can we account for the enormous variation in the percentage of correct responses? Dr. Everett, in a discussion of these and similar results, stresses the importance of "associative skills of interpretation."[34] That is precisely the point. When skills appear in *combinations,* as they usually do, they acquire new meanings which at first are not realized or appreciated. This fact is now recognized by progressive psychologists.[35] It is here that the ordinary stimulus-response psychology of learning breaks down hopelessly. *Nothing but insight can help us to use the old skills correctly in new or unusual situations.*

We may summarize our discussion as follows:

1) *All the skills of algebra must be made to depend on clearly formulated "principles."*
2) *These skills must be habituated both individually and collectively.*
3) *The composite use of these skills presupposes a conscious grasp of the basic principles and of their interrelations.*

7. *The Elimination of Obsolete or Useless Material.* In order to create time for real mastery, for motivation, and especially for functional thinking, the newer syllabi agree in eliminating certain topics or processes which formerly occupied a prominent place in the elementary algebra course. For an extensive discussion of this question, the reader may be referred to Thorndike's *Psychology of Algebra,* Chapter II, and to the National Report, Chapter V.

The guiding principle to be kept in mind in this connection has been formulated by the National Committee in the following manner:

"All topics, processes, and drill in technique which do not directly contribute to the development of the powers mentioned should be eliminated from the curriculum."

From either a practical or a cultural standpoint, there appears to be no justification for the retention of the following material in an introductory course:

1) *The involved manipulation of polynomial expressions.*[36]

[34] Everett, John P., *The Fundamental Skills of Algebra,* Chapter VI. Teachers College, Bureau of Publications, 1928.

[35] Bode, Boyd H., *Conflicting Psychologies of Learning,* Chapter XIV.

[36] Thus, Professor Thorndike suggests the elimination of the following: Addition and subtraction of long expressions; multiplication and division of polynomials; manipulation of fractions with polynomial terms; squares and cube root of polynomials resulting in answers of more than two terms; involved factorization of polynomials; reduction of radicals (or fractional exponents) whose index is greater than three; operations with polynomials containing radicals. (*The Psychology of Algebra,* p. 82.)

2) *"Nests" of parentheses.*
3) *Cases of factoring demanding the grouping of terms, or exceeding the three simple types suggested by the National Report.*
4) *Algebraic fractions containing trinomial denominators.*
5) *Types of equations which are very unlikely to arise in any genuine problem situation.*
6) *The "simplification" of radical expressions which are merely of manipulative interest.*
7) *The prescribed use of literal, fractional, and negative exponents.*

By applying such a pruning process we shall at last succeed in putting into bold relief the important objectives and shall create more time for worthwhile applications.

Only one obstacle, thus far, has held back the general acceptance of such a sane readjustment. It is the "vicious circle" created by the fact that the official examinations, until quite recently, have stressed the traditional program, and that textbooks have been built primarily for the passing of these examinations.[37] Without desiring to propose a Board of Censorship for the examining bodies, the writer feels that the official examinations should no longer be passively accepted by teachers, without careful scrutiny and critical discussion. The examining bodies should certainly be taken to task for any violation of progressive policies, for evidences of backwardness, and for an unwarranted and unfair distribution of emphasis. *Our examinations should be organs of reconstruction and progress, not of retardation and inhibition.* A public forum should be created in THE MATHEMATICS TEACHER, through the National Council of Teachers of Mathematics, for a systematic, annual appraisal of the most authoritative examinations. We may then hope for a type of algebra teaching which will proceed without fear and hesitation to the realization of the ideals that have been advocated so persistently and enthusiastically, by individuals and by committees, for nearly thirty years.

[37] The severe criticism of the College Entrance Examination Board, in the National Report, is at last beginning to bear fruit. Thus, the examination in "Elementary Algebra" given in June, 1929, allots but 40 credits to technique, while the remaining 60 credits involve "thought" questions. Above all, the manipulative exercises are chosen with some regard for relative values.

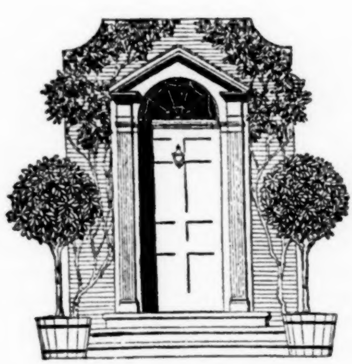
CONCLUSION

Any impartial and intelligent critic is almost certain to arrive at the following indictment against the prevailing type of algebra teaching:

1) *It is woefully inefficient.*
2) *It is constantly mistaking symbols for ideas, and manipulation for thinking.*
3) *It puts a premium on an inert and educationally meaningless technique, and neglects opportunities for application.*
4) *It fails to give prominence to the basic cultural ideas which alone justify a mandatory study of algebra.*

The erring subject of algebra, which has strayed so far from sanity, is at last finding itself at the crossroads. The genius of mathematics, in self-defense, will no longer permit it to evade the searching question, *Quo Vadis?*

We have tried to show that the dictates of common sense, of modern educational theory, and of sound mathematical training, demand a reorientation. *Whither Algebra?* Will it continue to travel on the road that leads to educational suicide, or will it establish its rightful place in the sun as one of the most significant and indispensable domains of human thought and endeavor?



NEWS NOTES

The Huntington (W.Va.) Mathematics Council was organized in 1926. They hold regular meetings and discuss such topics as "Some Interesting Facts About Early Greek Mathematicians," "The Use of the History of Mathematics in Teaching" and "A Contrast of the Dull and Bright Mind." Miss Catherine Watkins is chairman of the group and Miss Florence Oxley is secretary.

The mathematics section of the Central Association of Science and Mathematics Teachers held its annual meeting at the University of Chicago on Friday afternoon, November 29, 1929. The following program was given:

Appointment of Nominating Committee

Definition and Classification of Geometries
Professor Ernest P. Lane, The University of Chicago, Chicago, Ill.

Combined Courses in Plane and Solid Geometry
Professor Dunham Jackson, University of Minnesota, Minneapolis, Minn.

Method of Handling the Unit Plan of Teaching in Mathematics
Mr. O. M. Miller, Chicago Normal College, Chicago, Ill.

What Do You Want in the "Journal"?
Discussion led by Mathematics Dept. Editors.
C. N. Mills, Illinois State Normal University, Normal, Ill.
J. M. Kinney, Crane Junior College, Chicago, Ill.

General Discussion

Election of Officers

The New York Association of Teachers of Mathematics held a dinner meeting at the Men's Faculty Club of Columbia University on Saturday evening, January 11. The program consisted of a symposium on "Suggestions for Improving the Teaching of Mathematics in New York City." Superintendents in charge of Junior and Senior high schools were present as were principals of the junior and senior high schools. The association voted to send two representatives to the meeting of the National Council of Teachers of Mathematics at Atlantic City in February, one to represent the junior high schools and one the senior high schools.

The following paragraphs from the last bulletin (No. 2) of this association may be of interest:

"At the last meeting of the High School Teachers Association the president was authorized to appoint a committee for the purpose of investigating the reconstruction of the curricula in the high schools of New York City. This was decided upon in connection with a discussion of the high schools of the future. There is no doubt that, if this work is undertaken seriously, mathematics will be called upon to defend itself, not because mathematics in the high schools needs any defense, but because every subject will have to justify its place in the curriculum. Here is something for the Association of Teachers of Mathematics to look into. It is not enough that the chairman of departments take an active part in the current movements in the field of secondary mathematics. The classroom teacher must also become an

active participant in the work that is advocating the introduction of the best kind of mathematics for the pupils of today.

"All the pupils who come to the high schools and who want to take algebra and geometry must be permitted to do so. The content of these subjects is prescribed by a syllabus that provides for no differentiation in ability, although the pupils form an extremely heterogeneous group. Naturally, those pupils who find the work beyond them fail in large numbers. Hence, the teaching of algebra and of geometry, judged by "results" is poor. In fact, the semi-annual tables of statistics tell us that the results in algebra and geometry are poorer than in any other subject. Such a situation has been in existence in the city high schools for a number of years and is unfair to the pupils, to the teachers, and to the subject matter. Why should boys and girls be compelled to struggle unsuccessfully with subject matter that is too difficult for them? Why should teachers be compelled to struggle unsuccessfully with pupils who cannot possibly do any better with the work presented to them? The solution for this situation is to introduce modified courses of study both in algebra and in geometry for those pupils who cannot cope with the regular syllabus. From the administrative point of view this is entirely possible, as any capable program maker will admit. It is just as easy to provide for twelve grades of mathematics as for eight. The demand for this must come from the classroom teachers who are being accused of inefficiency, because so many of their pupils fail. Here is something for the Association of Teachers of Mathematics to work for in the Junior as well as in the Senior High Schools."

The most significant action taken at the business session of the Mathematics Section of The Illinois High School Conference in November was the adoption of the following resolution:

"Whereas, the National Council of Teachers of Mathematics furnishes an opportunity to write all teachers of mathematics and all organizations of mathematics teachers in a common purpose to improve the teaching of mathematics, and

"Whereas, we the Mathematics Section of the High School Conference, as the chief organization of mathematics teachers in the state of Illinois, desire to support the National Council in its work,

"Be it Resolved that we affiliate ourselves with the National Council and that henceforth the Mathematics Section of the High School Conference be known as the Illinois Branch of the National Council of Teachers of Mathematics."

The following officers were elected:

Chairman (1 year)—Mr. Edgar S. Leach, Evanston High School.

Vice-chairman (2 years)—Miss Lida Martin, Decatur High School.

Secretary-Treasurer (3 years)—Miss Gertrude Hendrix, University High School, Urbana, Illinois.

A collection amounting to $45.42 was taken, and turned over to the Director of the Conference for the purpose of helping to pay the expenses of Professor Slaught.

The work before the new Geometry Committee formed under the joint auspices of the Mathematical Association of America and the National Council of Teachers of Mathematics was presented by Mr. C. M. Austin of Oak Park, a member of the committee.

NEW BOOKS

Plane Geometry. By Charles Solomon and Herman H. Wright. New York, Charles Scribner's Sons, 1929. Pp. 340+viii. Price $1.40.

The book proposes "to present geometry in such a way as to enable the student to do his own thinking and to apply his knowledge to practical problems." Its aim is to train the student to discover what lines must be drawn and what plans must be followed in proving the various propositions, and thus put him "in the position of a discoverer." This attitude of a "discoverer" is to be brought about by a use of the analytic method. Each problem is preceded by an analysis or plan of solution or both; this is followed by a synthetic proof.

After reading the authors preface one would expect to find an exceptional arrangement, but there is little out of the ordinary about the book. The analysis referred to above is the most disappointing feature. It consists of statements, placed in italics between the "To Prove" and "Proof," which are not, in most cases, an analysis of the problem at all, but merely statements telling what the pupil must do in order to start the proof. Unless the teacher adds much to the material in the book the pupil will seldom feel like a "discoverer." Following this the proof is given in full, with practically all the reasons written out. Other reasons are given in article numbers or as Axiom 1, Axiom 2, and the like. In a very few cases a "Why?" is placed in the proof. The pupils will have little to do to get the correct proofs except look up a few article numbers.

There are over 1000 exercises in the book; they are wholly of the regular type, that is, separate originals based on propositions already proved. Many of them are grouped at the end of the Books; 75 at the end of Book I, 88 at the end of Book II, 146 at the end of Book III, 107 at the end of Book IV, 132 at the end of Book V. There are no exercises in related groups developing a certain idea.

At intervals through the book are short groups of new type questions on the immediately preceeding material. These are designated Group I, Group II, etc., and there are seventeen of them. These should prove suggestive to teachers, especially those who may not have had experience in making and giving the new type tests.

The book is well written and the exercises cover a wide range of material. It should be especially useful for schools that put emphasis on the applications of geometry. The figures are well drawn and set out in rectangles for emphasis.

Jas. H. Zant

Plane Trigonometry. By Carl A. Garabedian and Jean Winston. New York, McGraw-Hill Book Company, 1929. Pp. 306+xviii. Price $2.25.

It is evidently the purpose of the authors to have the student approach trigonometry "with the spirit of an investigator" and "thus be charmed and thrilled in much the same way as is the scholar exploring a new field." Evi-

dently the book is for "the student who can see beauty in trigonometry and take delight in developing the subject anew." They also assume that this can best be done by a natural and logical unfoldment of the subject matter, since, "it seems reasonable to suppose that this student—who can find interest in trigonometry *per se*—will prefer meeting the whole to being confronted by scattered fragments in some text on 'unified mathematics.'" The doubtful thing is, of course, the possibility of getting such pupils in our classes in sufficient numbers to warrant the use of a text especially built for them.

There are several extra-ordinary features to the book. Owing to the lack of space these cannot be discussed fully, but a few of them will be mentioned briefly. The first one noticed is the syllabus form of arrangement. It is described in the preface as "a comprehensive set of lecture notes." It is a strictly logical arrangement of the subject matter and is full, and beautifully coordinated—for one who is already familiar with the subject matter. The average student will fail utterly to see this logic, unless he reviews the book carefully at the end of the course. One would hesitate to predict the outcome of using such an arrangement in the classroom, since it is so different from anything ordinarily used. It is doubtful if the student will be able to understand the statement made until the instructor has lectured and illustrated rather carefully.

Other things worth noting are the chapter titles and contents. Chapter I, The General Angle and the Trigonometric Functions; Chapter II, Variation of the Functions; Chapter III, Five Place Tables of Natural Functions. Solution of Right Triangles; Chapter IV, Graphs of Functions; Chapter V, Logarithms and Solution of Right Triangles; Chapter VI, Relations Between Functions of a Single Angle. Identities and Equations; Chapter VII, Additional Theorems and Derived Relations; Chapter VIII, Solution of Oblique Triangles. A glance at the order of these chapters will show some variation out of the ordinary. In the first place the general angle is defined first and the functions defined with reference to it. There is a well illustrated chapter on the graphs of the functions. Logarithms and the solution of right triangles begin on the 124th page, practically half way through the book.

The contents of some of the chapters are also interesting. Chapter III presents the solution of the right triangle by the natural functions as carefully as it is ordinarily taught by logarithms. According to the authors, the increasing importance of computing machines has made this highly desirable. This is accomplished by an adequate discussion of accuracy and the introduction of "rounded" processes of multiplication and division. Chapter IV is a discussion of the graphs of the functions, both from the standpoint of their line representations and with Cartesian Coordinates as algebraic equations. Plates of the actual graphs are included, those of the simple curves as well as those of sine and cosine curves of modified period and amplitude, (as $y = \sin\frac{1}{2}\ x$, $y = \frac{1}{2}\sin x$, and the like), and synthesis of sine and cosine curves, (as, $y = \cos x + \sin x$, $y = \sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x$, and the like). This chapter should add much to the cultural values of the course as well as to give more meaning to the values of the functions for the various sized angles.

From the standpoint of subject matter and scholarship, the book is ex-

ceedingly well written. It may be considered a complete treatment of trigonometry from the elementary point of view. It should be an excellent reference book for a high school mathematics teacher and for classes studying mathematics in college and high school. It is especially adaptable to this kind of work, since it is written in the syllabus form. No tables are included in the copy reviewed and evidently none are intended, since reference is made in Chapter III to The Macmillan Logarithmic and Trigonometric Tables prepared by E. R. Hedrick, The Macmillan Company, 1926.

The book is extremely well printed and attractive. All the important words and phrases are set in bold faced type, which makes it very easy to follow a certain discussion or to find the meanings of certain terms when using the book for reference, either for review work or for other courses. The figures and exercises are numbered according to the articles, thus, if three figures are used in Article 44, they are numbered 44.1, 44.2 and 44.3. The articles are numbered according to chapters, that is, those in Chapter I are 11, 12, 13,.... 19; in Chapter II, 21, 22, 23,....25; in Chapter VIII, 81, 82,....86. Thus Article 56 means the sixth article in Chapter V, etc. These features should also be especially helpful in reference work.

JAS. H. ZANT

College Algebra. N. J. Lennes. Harper and Brothers. 1928. xiv+301 pages. Price $2.25.

In addition to its attractive appearance and careful paging, College Algebra has the following features:

1. The topics are arranged in as logical a sequence as is possible so that the students will not feel that they are being given a series of disconnected chapters.
2. The historical sketch in Chapter XXIII is very much worthwhile. Pupils who reach college algebra ought to become acquainted with a little of the history of the elementary mathematics. Whether or not it would have been better to have inserted the various paragraphs in connection with the subject-matter throughout the book is an unsettled question. Other books have done that. The tendency on the part of the student has been, however, to skip such paragraphs. Will the instructors refer their classes to this material at the end of the book?
3. The chapter on Algebraic Exercises from Analytic Geometry and Calculus ought to prove interesting to enterprising students. Its value in general, however, is doubtful. Is this chapter introduced to make the book modern? It would have been better in that case to have introduced the various topics from the Analytic Geometry and the Calculus in conjunction with the algebra throughout the text.
4. A review of the elementary algebra is a worthwhile portion of a text in college algebra; and the review in this book is well-balanced. However, many of the exercises appear far from elementary when one considers the make-up of the elementary texts of today. This is not necessarily an adverse criticism of this book. It is a difficulty with which teachers will have to cope who have occasion to use the early part of the book for review.
5. Each chapter begins with an introductory sentence or paragraph telling briefly of the topic which the chapter treats. Theoretically this

is an excellent innovation. However, to what extent does the average college freshman or high school senior (for this text might well be used with some of the high school advanced algebra classes) refer to other paragraphs than those which enable him to solve the exercises and problems.

In general, the text is a distinct contribution to the teaching of college algebra.

JOSEPH B. ORLEANS
George Washington High School
New York City

Plane Trigonometry. N. J. Lennes and A. S. Merrill. Harper and Brothers. 1928. x+207+92 pages. Price $2.20 with tables. Without tables $1.60. Tables alone $1.20.

This text, like the College Algebra, is very attractive and should appeal to the students because of its make-up. It begins with functions of acute angles and the solution of the right triangle, first by natural functions and then by logarithms. This covers the first 56 pages. The functions of any angle and of the sums and differences bring the work to page 70. Then follow the solution of the oblique triangle from page 71 to page 90, variation of trigonometric functions to page 110, inverse functions and trigonometric equations to page 124. The rest of the book (63 pages) deals with De Moivre's Theorem, Exponential and Hyperbolic Functions, the Slide Rule, Applications of Logarithms to problems in Business, Surveying and Physics, a chapter on Spherical Trigonometry and a Historical Sketch.

There is an abundance of exercises and problems that furnish plenty of good drill material and, in addition, eight cumulative reviews.

It might have been well to have introduced the solution of the oblique triangle earlier in the book, so that the student need not wait until he has completed the study of the functions of any angle. There is no reason why the two cannot run together, thus giving the classes more time with the exercises on triangles. The explanatory material in the chapter on the oblique triangle seems to be crowded into a few pages with miscellaneous exercises following at the end, It is far better to separate the different cases, so that the student does not see too much at one time.

The book is well adapted for group work in a class that has some exceptional students because of the supplementary material that it contains. These extra topics should prove worthwhile also to ordinary groups of students. They would probably mean more to them than some of the abstract work of the trigonometry itself.

JOSEPH B. ORLEANS
George Washington High School
New York City

The National Council
of
Teachers of Mathematics

Organized 1920 Incorporated 1928

This is to Certify
that

Chicago Men's Mathematics Club

has affiliated with the
National Council of Teachers of Mathematics
and is officially enrolled as an
Associated Organization

Dated: June 6, 1929.

No. 1.

Edwin W. Schreiber
Secretary

Facsimile of Certificate No. 1 made out in favor of the Chicago Men's Mathematics Club, the birth-place of the National Council. Has your club received one of these beautiful engraved certificates? Write your Secretary, Edwin W. Schreiber, 434 W. Adams St., Macomb, Illinois, for further particulars.